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CONTENTS OF VOLUME 86

NUMBER 1, APRIL 1981

NUMBER 2, MAY 1981

NUMBER 3, JUNE 1981

The Subject Index for Volume 86 will appear in the December 1981 issue as part of a cumulative index for the year 1981.

RADIATION RESEARCH 86, 264–276 (1981)

Proximity Functions for General Right Cylinders[1]

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Distributions of distances between pairs of points within geometrical objects, or the closely related proximity functions and geometric reduction factors, have applications to dosimetric and microdosimetric calculations. For convex bodies these functions are linked to the chord-length distributions that result from random intersections by straight lines. A synopsis of the most important relations is given. The proximity functions and related functions are derived for right cylinders with arbitrary cross sections. The solution utilizes the fact that the squares of the distances between two random points are sums of independently distributed squares of distances parallel and perpendicular to the axis of the cylinder. Analogous formulas are derived for the proximity functions or geometric reduction factors for a cylinder relative to a point. This requires only a minor modification of the solution.

1. INTRODUCTION

The distributions of distance between pairs of points within geometrical objects were first utilized by Berger (*1*) in dosimetric computations. These point-pair distributions have broad applicability in calculations of absorbed dose from radionuclides [see (*2–11*)]; they are also relevant to microdosimetry (*12–14*). Analytical expressions can be given for configurations such as spheres, slabs, or spherical shells. A solution for cylinders that contains one quadrature without singularities is derived in the present article. As in an earlier article dealing with chord-length distributions (*15*), the solution will be obtained for cylinders with arbitrary cross section; the formula for circular cylinders results as a special case.

The result is applicable to calculations of absorbed doses with cylindrical sources or receptors. Because of the utilization of cylindrical detectors the solution is also relevant to microdosimetry. The accompanying article (*16*) uses the results of the present study for an assessment of the degree of equivalence achievable between spherical and cylindrical microdosimetric detectors.

The distance distribution of a geometrical object is essentially equivalent to two other concepts, the proximity function and the geometric reduction factor. The interrelations between the three concepts are given in Section 2.1; Sections 2.2 and 2.3 deal with the connection to the chord-length distributions that result when the geometrical body is randomly intercepted by straight lines. Readers interested only in the solution for cylinders may first ignore Sections 2.2 and 2.3, but may consult them for equations required in practical applications.

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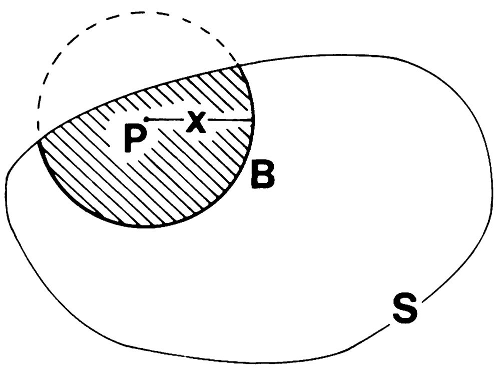

FIG. 1. Diagram illustrating the definition of the proximity function of a site, S. P is a random point in S. The integral proximity function, $S(x)$, equals the expected volume represented by the shaded region; the differential proximity function, $s(x)$, equals the expected surface indicated by the circular line segment B; and the geometric reduction factor, $U(x)$, equals the ratio of B to the total surface of the sphere.

For brevity, various considerations in this article will refer to only one of the related concepts, for example, the proximity function. It should be realized that reference could equally be made to the other concepts.

2. PROXIMITY FUNCTIONS AND SIMILAR CONCEPTS AND THEIR RELATIONS TO THE CHORD-LENGTH DISTRIBUTIONS

2.1 Proximity Functions, Distance Distribution, and Geometric Reduction Factor

The integral proximity function, $S(x)$, of a region S is equal to the expected volume of the region that is contained in a sphere of radius x centered at a random point of S. The differential proximity function $s(x)$ is the derivative of $S(x)$, i.e., $s(x)dx$ is the expected volume of S contained in a spherical shell of radius x and thickness dx that is centered at a random point of S. These notions are indicated schematically in Fig. 1.

Dividing $s(x)$ by the volume, V, of S one obtains, as can be shown (*12*), the density of distances between pairs of random points in S (see Fig. 1). Berger (*1*) had earlier termed this the "pair distance distribution," $p(x)$:

$$p(x) = s(x)/V. \tag{1}$$

The proximity functions or distance distributions can also be defined for surfaces or linear structures in three-dimensional space, R_3. Volume is then replaced by surface or length. Since such structures may be contained in one- or two-dimensional linear subspaces the case of general dimensionality is of interest. The subsequent formulas in this section will therefore apply to arbitrary dimensions; where this is not the case separate relations will be quoted for three-dimensional space, R_3, and two-dimensional space, R_2. V and S designate volume and surface in R_3, and A and C designate area and circumference in R_2.

The function $p(x)$ has the advantage that it is a properly normalized probability distribution; the nonnormalized function, $s(x)$, on the other hand, is more generally

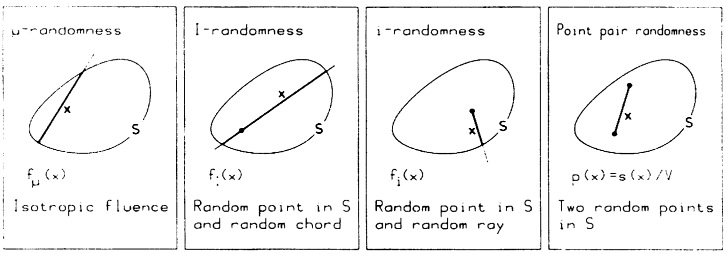

FIG. 2. Diagram indicating the nature of μ-, I-, and i-randomnesses, and of point-pair randomness.

applicable, because it exists also for unbounded structures, such as infinite lines or areas or infinite cylinders.

At small values of x the proximity function of a volume goes toward $4\pi x^2$ and that of an area toward $2\pi x$. A related quantity that converges toward 1 at $x = 0$ can be more practical in numerical applications; it will be used interchangeably with $s(x)$ or $p(x)$:

$$\begin{aligned} U(x) &= s(x)/4\pi x^2 = p(x)V/4\pi x^2 \qquad (\text{in } R_3) \\ &= s(x)/2\pi x = p(x)A/2\pi x \qquad (\text{in } R_2). \end{aligned} \tag{2}$$

This quantity has been termed the geometric reduction factor by Berger (*1*), and it is frequently used in dose calculations for internal emitters.[2] If a spherical shell of radius x is centered at a random point of S, then $U(x)$ is equal to the average fraction of this shell that lies within S (see Fig. 1).

2.2 Chord-Length Distributions

Chord-length distributions result when geometric configurations are randomly intercepted by straight lines. There are different modes of randomness that lead to different distributions of chord length (*15, 17, 18*). Three important types that are related to each other and are also linked to the proximity functions are indicated in Fig. 2.

The condition where a site S is exposed to a uniform, isotropic fluence of straight infinite random lines has been termed μ-randomness (*17*). A second condition, I-randomness (interior radiator randomness), results if random points are chosen within S and straight lines are laid through these points with random orientation (*17*). i-randomness results from the same condition if rays originate from the random points (*15*). The distribution $p(x)$ of distance between two random points in S is indicated in the last panel of Fig. 2.

The probability densities of the intercepts, x, for the different types of randomness are designated by $f_\mu(x)$, $f_\mathrm{I}(x)$, and $f_\mathrm{i}(x)$. The sum distributions—for convenience summed from the right—are designated by $F_\mu(x)$, $F_\mathrm{I}(x)$, and $F_\mathrm{i}(x)$. The mean values are designated by $\bar{x}_\mu$, $\bar{x}_\mathrm{I}$, and $\bar{x}_\mathrm{i}$. For example,

[2] The quantity is commonly called average geometric reduction factor $\Psi_a(x)$, and a related concept (see Section 4) is called geometric reduction factor $\Psi(x)$. A different symbol, $U(x)$ is chosen here to avoid confusion with energy fluence (*19*).

$$F_\mu(x) = \int_x^\infty f_\mu(s)ds \qquad \text{and} \qquad \bar{x}_\mu = \int_0^\infty x f_\mu(x)dx = \int_0^\infty F_\mu(x)dx. \tag{3}$$

The next section is a condensed summary of essential interrelations between the different functions.

2.3 Relations between the Proximity Function and the Chord-Length Distributions

Kingman (*18*) has given the important relation between the chord-length distributions for I- and μ-randomnesses:

$$f_\mathrm{I}(x) = x f_\mu(x)/\bar{x}_\mu. \tag{4}$$

A somewhat more complicated relation holds for i-randomness (*15*):

$$f_\mathrm{i}(x) = F_\mu(x)/\bar{x}_\mu. \tag{5}$$

Finally one obtains for convex sites:

$$U(x) = F_\mathrm{i}(x) = \int_x^\infty F_\mu(s)ds/\bar{x}_\mu. \tag{6}$$

The relation holds because a random shift x of a random point in a convex body S will lead with probability $F_\mathrm{i}(x)$ to a point still in S. In R_3 this probability is equal to the fraction, $U(x)$, of a spherical surface of radius x that is contained in S, if the shell is centered at P. In R_2 an analogous argument applies.

The separate concepts $F_\mathrm{i}(x)$ and $U(x)$ are required, because $F_\mathrm{i}(x)$ and $U(x)$ differ for nonconvex structures.

By using Eqs. (4–6) one can also relate the chord-length densities to the derivatives of the geometric reduction factor of convex sites:

$$-U'(x) = f_\mathrm{i}(x) \qquad [-U'(0) = f_\mathrm{i}(0) = 1/\bar{x}_\mu \text{ (see Eq. (5))}]; \tag{7}$$

$$U''(x) = f_\mu(x)/\bar{x}_\mu = f_\mathrm{I}(x)/x, \tag{8}$$

where the mean chord length, $\bar{x}_\mu$, is given by the Cauchy theorem that applies to convex sites [see (*18*)]:

$$\begin{aligned} \bar{x}_\mu &= 4V/S \qquad (\text{in } R_3) \\ &= \pi A/C \qquad (\text{in } R_2). \end{aligned} \tag{9}$$

From Eqs. (4–6) one obtains by partial integration the relations between the moments ($n = 0, 1, 2 \ldots$):

$$\overline{x_\mu^{n+2}}/(n+1)(n+2)\bar{x}_\mu = \overline{x_\mathrm{I}^{n+1}}/(n+1)(n+2) = \overline{x_\mathrm{i}^{n+1}}/(n+1) = \int_0^\infty x^n U(x)dx \tag{10}$$

$$\begin{aligned} &= \overline{x_p^{n-2}}V/4\pi \qquad (\text{in } R_3) \\ &= \overline{x_p^{n-1}}A/2\pi \qquad (\text{in } R_2) \end{aligned} \tag{11}$$

The indices μ, I, i, and p refer to the densities $f_\mu(x)$, $f_\mathrm{I}(x)$, $f_\mathrm{i}(x)$, and $p(x)$. For R_3 and $n = 0$ to 4 these important relations are listed explicitly in Section 3 [see Eqs. (17–21)].

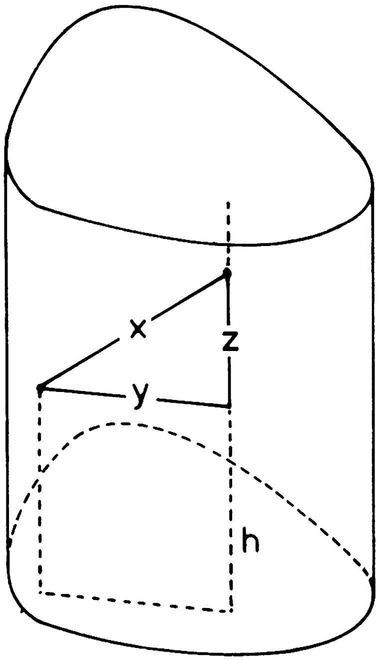

FIG. 3. Diagram illustrating the computation of the distribution of distances, x, between two random points in a cylinder in terms of the independent random variables y and z.

Although it is of no direct importance in the present context, one may note the striking fact that the third moment for I- or i-randomness is independent of the shape of a convex body in R_3:

$$\overline{x_I^3} = \overline{4x_i^3} = 3V/\pi, \tag{12}$$

while an analogous relation holds in R_2:

$$\overline{x_I^2} = \overline{3x_i^2} = 3A/\pi. \tag{13}$$

This concludes the general considerations. The subsequent section gives solutions for cylinders.

3. PROXIMITY FUNCTIONS FOR CYLINDERS

A formula requiring a numerical integration was derived previously (*15*, *20*) for the chord-length distributions $F_\mu(x)$ of general cylinders. By a further integration one could, according to Eq. (6), obtain the geometric reduction factor or the proximity function. A disadvantage of this procedure is that the integrals contain various singularities. The functions $s(x)$ or $U(x)$ are, however, considerably simpler than the complicated chord-length distributions for μ-randomness. In fact, there is as indicated in Fig. 3 a direct solution that constructs the distribution of point-pair distances, x, for the cylinder from the distribution of distances, y, perpendicular to the axis of the cylinder (horizontal distances) and the distribution of distances, z, in the direction of the axis (vertical distances). The method utilizes the fact that x^2 is the sum of y^2 and z^2, and the horizontal and vertical distances y and z are independent random variables.

The formal derivation is given separately in the Appendix. One obtains the following equation for the proximity function of a right cylinder of height h and with arbitrary cross section:

$$s(x) = 2x \int_{z_1}^{z_2} \left(1 - \frac{z}{h}\right) \frac{s_c((x^2 - z^2)^{1/2})}{(x^2 - z^2)^{1/2}} dz; \tag{14}$$

with

$$z_1 = (\text{Max}\ (0, x^2 - d^2))^{1/2}, \qquad z_2 = \text{Min}\ (x, h), \qquad x \le (h^2 + d^2)^{1/2}, \tag{15}$$

where $s_c(y)$ is the proximity function of the cross section of the cylinder, and d is the diameter (i.e., maximum width) of the cross section.

The equation in this general form is the essential result of this article. To use the result for complicated cross sections one needs to derive $s_c(y)$ numerically; this may require separate integrations or Monte Carlo methods.

For a circle and a rectangle analytical expressions of $s_c(y)$ are listed in the Appendix. For a circular cylinder of diameter d one obtains with Eq. (A.6)

$$s(x) = 8x \int_{z_1}^{z_2} \left(1 - \frac{z}{h}\right) [\cos^{-1}((x^2 - z^2)^{1/2}/d) - ((x^2 - z^2)(d^2 - (x^2 - z^2)))^{1/2}/d^2] dz. \tag{16}$$

Corresponding equations hold for the pair-distance density, $p(x) = 4s(x)/(hd^2\pi)$, or the quantity $U(x) = s(x)/(4\pi x^2)$.

The integrals in Eq. (14) or (16) are readily evaluated since they contain no singularities. Figure 4 represents solutions $U(x)$ for various values of the elongation, h/d, of the cylinder. The function $U(x)$ is plotted instead of $s(x)$ because this permits higher accuracy at small values of x.

Equations (10) and (11) for the moments can be used to check the numerical accuracy of the results [see Eqs. (19) and (22)] and to derive the mean values $\bar{x}_i$, $\bar{x}_1$, and $\bar{x}_p$ [see Eqs. (17) and (20)]. These mean values are plotted in Fig. 5 together with the mean chord length for μ-randomness, $\bar{x}_\mu = dh/(d/2 + h) = -1/U'(0)$.

Separately listed for $n = 0$ to 4, and with $c = V/4\pi$, the relations from Eqs. (10) and (11) have the form:

$$\int U(x)dx = c\overline{x_p^{-2}} = \overline{x_i} = \bar{x}_1/2 = \overline{x_\mu^2}/2x_\mu, \tag{17}$$

$$\int xU(x)dx = c\overline{x_p^{-1}} = \overline{x_i^2}/2 = \overline{x_1^2}/6 = \overline{x_\mu^3}/6x_\mu, \tag{18}$$

$$\int x^2U(x)dx = c = \overline{x_i^3}/3 = \overline{x_1^3}/12 = \overline{x_\mu^4}/12x_\mu, \tag{19}$$

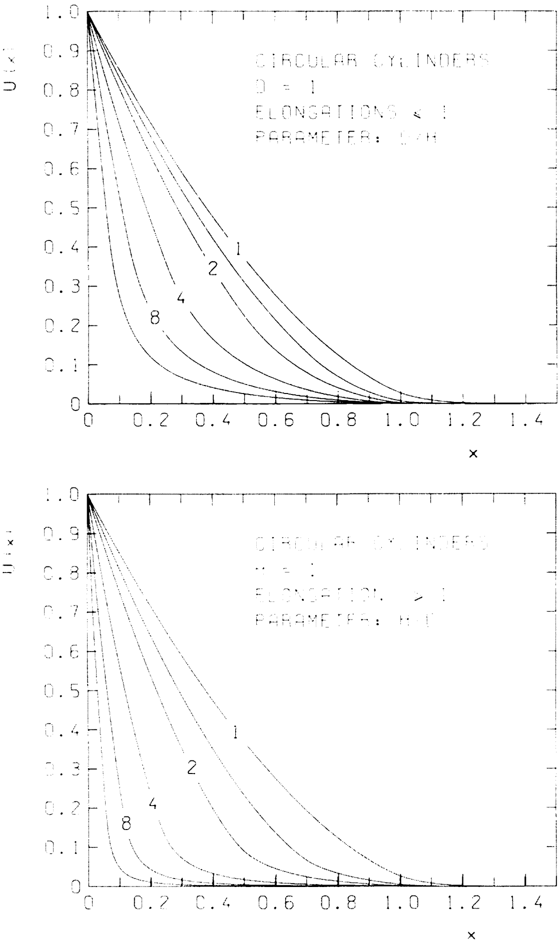

FIG. 4. The geometrical reduction factors, $U(x)$, for right circular cylinders of elongations 1/16, 1/8, 1/4, 1/2, $1/(2)^{1/2}$, 1, $2^{1/2}$, 2, 4, 8, and 16. The larger axis is taken to be of unit length, and the ratio of the larger to the smaller axis is given as parameter with some of the functions. The differential proximity function $s(x)$ is equal to $4\pi x^2 U(x)$.

$$\int x^3 U(x)dx = cx_p = \overline{x_i^4}/4 = \overline{x_I^4}/20 = \overline{x_\mu^5}/20x_\mu, \tag{20}$$

$$\int x^4 U(x)dx = c\overline{x_p^2} = \overline{x_i^5}/5 = \overline{x_I^5}/30 = \overline{x_\mu^6}/30x_\mu \tag{21}$$

$$= c(d^2/4 + h^2/6). \tag{22}$$

The integrals run from 0 to the maximum value of x. Equations (17) to (21) hold generally for convex bodies in R_3. Equation (22) is restricted to circular cylinders; it is based on the fact that $\overline{x_p^2}$ is the sum of the mean squared distance, $d^2/4$, for point pairs inside a circle and the mean squared distance, $h^2/6$, for point pairs on a line segment.

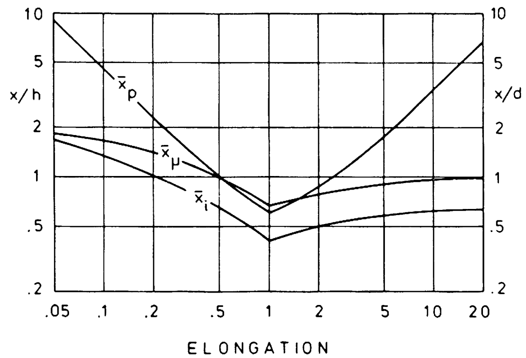

FIG. 5. Mean chord length, $\bar{x}_\mu$, for μ-randomness, mean chord length, $\bar{x}_i$, for i-randomness, and mean distance, $\bar{x}_p$, between two random points for circular cylinders of various elongations. The values are given relative to the length of the smaller axis. The mean chord length, $\bar{x}_I$, for I-randomness is equal to $2\bar{x}_i$ and is therefore not plotted.

In certain applications it is practical to utilize the limiting form of the solution for very long and for very flat cylinders. For long cylinders ($h \gg d$) and for moderate values of x one can use the limiting form of Eq. (16) for infinite height; for large values of x one can disregard the radial extension of the cylinder. With these two approximations Eq. (16) reduces to

$$\begin{aligned} s(x) &= 8x \int_{z_1}^{x} [\cos^{-1} ((x^2 - z^2)^{1/2}/d) \\ &\qquad - ((x^2 - z^2)(d^2 - (x^2 - z^2)))^{1/2}/d^2]dz, && \text{for } x \ll h \\ &= d^2\pi(1 - x/h)/2 && \text{for } x \gg d, \quad (23) \end{aligned}$$

with the limit values

$$\bar{x}_\mu = d; \qquad \bar{x}_i \approx 0.662d; \qquad \bar{x}_p = h/3. \qquad (24)$$

For flat cylinders ($h \ll d$) and for moderate values of x one can use formulas for the infinite slab; for large x one can disregard the vertical extension and use, with inclusion of the factor h, the formula for the disk [see Eq. (A.6)]. This leads to

$$\begin{aligned} s(x) &= 4\pi x^2(1 - x/2h) && \text{for } x < h \\ &= 2\pi hx && \text{for } h < x \ll d \\ &= 4hx\left(\cos^{-1}\left(\frac{x}{d}\right) - \frac{x}{d^2}(d^2 - x^2)^{1/2}\right) && \text{for } h \ll x < d, \end{aligned} \qquad (25)$$

with the limit values

$$\bar{x}_\mu = 2h; \qquad \bar{x}_i \approx (\ln (d/h) + 0.3069)\cdot h/2 \text{ (see (20))};$$
$$\bar{x}_p = 64d/45\pi \approx 0.4527d. \qquad (26)$$

The solution for a right cylinder with square cross section, i.e., a rectangular parallelepiped, is obtained by inserting Eq. (A.7) into Eq. (14). This leads to a

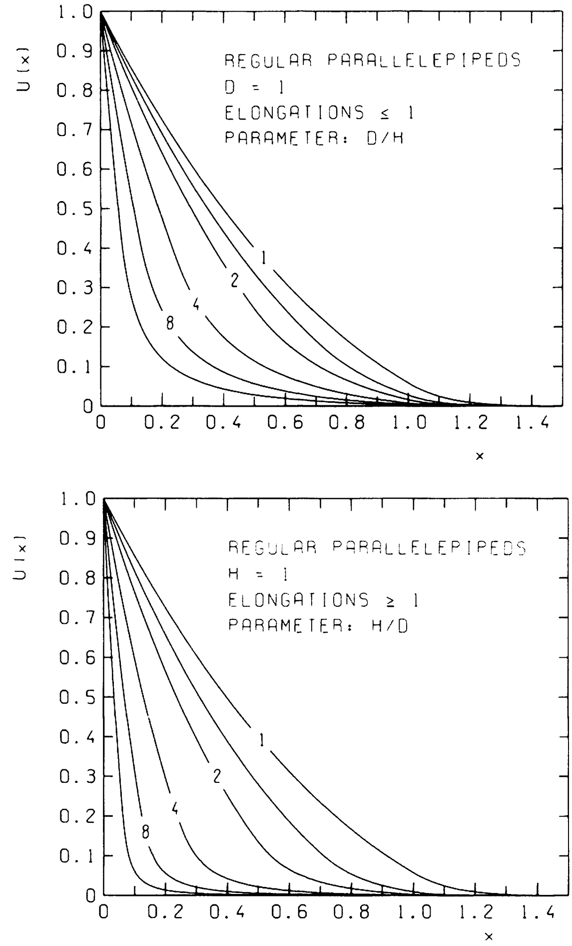

FIG. 6. The geometrical reduction factors, $U(x)$, for regular parallelepipeds with two sides equal to d and one side equal to h. Curves are given for elongations h/d = 1/16, 1/8, 1/4, 1/2, $1/(2)^{1/2}$, 1, $2^{1/2}$, 2, 4, 8, and 16. The larger side is taken to be of unit length. The ratio of the larger to the smaller side is given as parameter with some of the functions. The differential proximity function, $s(x)$, is equal to $4\pi x^2 U(x)$.

more complicated expression than Eq. (16), but the integration is also straightforward without singularities. Figure 6 gives the resulting functions $U(x)$.

4. PROXIMITY FUNCTIONS OF A CYLINDER RELATIVE TO A POINT

4.1 Generalized Definition of the Proximity Function and Related Concepts

The pair distance distribution, $p(x)$, and the proximity function, $s(x)$, relate to the distances between pairs of points picked at random in the specified region. Berger previously pointed out (*1*) that an entirely similar procedure, using a pair distribution function, can also be applied in considerations of the transfer of energy from a source region to any other region in the medium. This notion is closely related to concepts such as the absorbed energy fraction that is used in dose calculations for internal emitters (*1–11*). One can define a proximity function of a region S (the source region) relative to a region R (the receptor region). The integral proximity function $S_{RS}(x)$ of S relative to R is the expected volume of S that is contained within a distance up to x from a random point in R. The differential proximity function $s_{RS}(x)$ is the derivative of $S_{RS}(x)$.

The proximity function $s_{RS}(x)$ is equal to the pair-distance distribution $p_{RS}(x)$ for random points picked in R and S, multiplied by the volume V_S of the source region:

$$s_{RS}(x) = V_S \cdot p_{RS}(x) = V_S/V_R \cdot s_{SR}(x). \tag{27}$$

The nonnormalized function is required because it is applicable also to unbounded source regions.

It is again practical to introduce the geometrical reduction factor

$$U_{RS}(x) = s_{RS}(x)/4\pi x^2 = V_S/V_R \cdot U_{SR}(x). \tag{28}$$

$U_{RS}(x)$ is the probability that a random displacement of magnitude x from a random point in R leads to a point in S. This is equal to the expected fraction of a spherical surface of radius x that is contained in S, if the sphere is centered at a random point of R.

4.2 Solutions for the Cylinder

Of special importance for the calculation of absorbed dose in and around extended sources is the simple case where the receptor region is a single point R. Various solutions for this case have been obtained by Berger (*1*), among them the one for infinite cylinders. In the following, the solution for finite cylinders will be given. The derivation requires only slight modifications of the solution utilized in Section 3; it is also relegated to the Appendix.

For simplicity, the indices of the functions $s_{RS}(x)$ and $U_{RS}(x)$ will be omitted in the remainder of this section; it will be understood that the functions refer to the cylindrical source region and to a point of specified location. For easier reference $s(x)$ can be called a *point proximity function* of S.

The position of the point R will (in addition to suitable horizontal coordinates) be specified by its vertical distance b from the face of the cylinder and away from the cylinder. To simplify the formulas only nonnegative values of b will be considered. For points between the two planes through the faces of the cylinder ($b < 0$) the solution can evidently be expressed as the sum of two solutions with $b = 0$.

As shown in the Appendix, one obtains the following point proximity function for the right cylinder with arbitrary cross section and with height h:

$$s(x) = x \int_{z_1}^{z_2} \frac{s_c((x^2 - z^2)^{1/2})}{(x^2 - z^2)^{1/2}} dz, \tag{29}$$

with

$$z_1 = \text{Max}\,(b, (\text{Max}\,(0, x^2 - y_2^2))^{1/2}) \text{ and } z_2 = \text{Min}\,(b + h, (x^2 - y_1^2)^{1/2}) \tag{30}$$

and

$$(y_1^2 + b^2)^{1/2} < x < ((b + h)^2 + y_2^2)^{1/2}. \tag{31}$$

$s_c(y)$ is the point proximity function of the cross section of the cylinder relative to the reference point, R; y_1 and y_2 are the minimum and the maximum of y.

In the special case of a circular cylinder of radius r one obtains with Eq. (A.9) from the Appendix and with the distance a from the axis of the cylinder

$$s(x) = 2x \int_{z_1}^{z_2} \cos^{-1} \left(\mathrm{Max} \left(-1, \frac{x^2 - z^2 + a^2 - r^2}{2a(x^2 - z^2)^{1/2}} \right) \right) dz. \tag{32}$$

Equations (30) and (31) hold with $y_1 = \mathrm{Max}\ (0, a - r)$ and $y_2 = a + r$.

The solution does not apply for $a = 0$; in this case one obtains from Eq. (29), with $s_c(y) = 2\pi y$ for $y \leq r$:

$$s(x) = 2\pi x(z_2 - z_1), \qquad a < x < ((a + h)^2 + r^2)^{1/2}. \tag{33}$$

The geometric reduction factor, $U(x) = s(x)/4\pi x^2$, commonly designated by $\Psi(x)$, has previously been given for the special case of infinite circular cylinders (*1*).

APPENDIX: DERIVATION OF THE SOLUTION FOR RIGHT CYLINDERS

1. Density of $x = (y^2 + z^2)^{1/2}$ *from Independent Densities of* y *and* z

Let x be the distance between two random points in the cylinder, and y and z the horizontal and vertical distances. Then y and z are independently distributed, and $x^2 = y^2 + z^2$.

As a first step a general expression for the density $p(x)$ as a function of the densities $p_1(z)$ and $p_2(y)$ of y and z will be derived. Insertion of actual expressions for $p_1(z)$ and $p_2(y)$ will be a second step.

To make the derivation more transparent, it is helpful to introduce separate symbols, $X = x^2$, $Y = y^2$, and $Z = z^2$, for the squares of the random variables and also separate symbols, $\pi(X)$, $\pi_1(Z)$, and $\pi_2(Y)$, for the densities of these squares. Because of the additivity, $X = Y + Z$, and the independence of Y and Z one has the familiar convolution relation

$$\pi(X) = \int_0^X \pi_2(X - Z)\pi_1(Z)dZ. \tag{A.1}$$

The relation between the density of the random variable x and the density of its square X is

$$\pi(X) = p(x)\frac{dx}{dX} = p(x)/2x; \tag{A.2}$$

analogous relations hold for $\pi_1(Z)$ and $\pi_2(Y)$.

By inserting these relations and $dZ = 2zdz$ into Eq. (A.1) one obtains

$$p(x)/2x = \int_0^x \frac{p_2((x^2 - z^2)^{1/2})}{2(x^2 - z^2)^{1/2}} \cdot \frac{p_1(z)}{2z} 2zdz \tag{A.3}$$

and therefore

$$p(x) = x\int_0^x \frac{p_2((x^2 - z^2)^{1/2})p_1(z)}{(x^2 - z^2)^{1/2}} dz. \tag{A.4}$$

2. Proximity Function for the Cylinder

One readily obtains the distance distribution for a line segment of length h:

$$p_1(z) = 2(1 - z/h)/h. \qquad \text{(A.5)}$$

By inserting this into Eq. (A.4) and switching from $p(x)$ and $p_2(y)$ to the proximity functions one obtains the general solution, Eq. (14).

The proximity function for a circular surface of diameter d is (*17*)

$$s(x) = 4x\left(\cos^{-1}\left(\frac{x}{d}\right) - \frac{x}{d^2}(d^2 - x^2)^{1/2}\right), \qquad x \leq d. \qquad \text{(A.6)}$$

This together with Eq. (14) leads to the solution, Eq. (16), for circular cylinders.

The proximity function for a square of side length d is somewhat more complicated [see also (*18*) for the general case of a rectangle]:

$$s(x) = 2x \begin{cases} \dfrac{x^2}{d^2} - \dfrac{4x}{d} + \pi, & x \leq d, \\[2ex] \pi - 2 - 4\cos^{-1}\left(\dfrac{d}{x}\right) + 4\left(\dfrac{x^2}{d^2} - 1\right)^{1/2} - \dfrac{x^2}{d^2}, & d \leq x \leq 2^{1/2}d. \end{cases} \qquad \text{(A.7)}$$

The formula for a regular parallelepiped is therefore not given in explicit form. However, the numerical integration of Eq. (14) with Eq. (A.7) is readily performed and contains no singularities; the solutions are given in Fig. 6.

3. Solution for a Cylinder Relative to a Point

With the coordinate b (≥ 0), as defined in Section 4, one obtains the distribution of vertical distances from the point R to the cylinder:

$$p_1(z) = 1/h \qquad \text{for} \qquad b \leq z \leq b + h. \qquad \text{(A.8)}$$

By inserting this into Eq. (A.4) and switching to the point proximity functions, one obtains the solution, Eq. (29), for the general cylinder.

For a circular cross section with radius r and for the distance a of the point from the center one obtains

$$s_c(y) = 2y\cos^{-1}\left(\text{Max}\left(-1, \frac{y^2 + a^2 - r^2}{2ya}\right)\right); \ \text{Max}\,(a - r, 0) \leq y \leq a + r. \qquad \text{(A.9)}$$

By inserting this into Eq. (29) one obtains the solution, Eq. (32), for the circular cylinder.

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