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Calculus of Variations

Existence and nonexistence in the liquid drop model

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Abstract

We revisit the liquid drop model with a general Riesz potential. Our new result is the existence of minimizers for the conjectured optimal range of parameters. We also prove a conditional uniqueness of minimizers and a nonexistence result for heavy nuclei.

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1 Introduction

Let $N \geq 2$, $\lambda \in (0, N)$ and $m > 0$ (λ and m are not necessarily integers). For any measurable set $\Omega \subset \mathbb{R}^N$, define

$$\mathcal{E}(\Omega) = \operatorname{Per}\Omega + D(\Omega), \quad D(\Omega) = \frac{1}{2}\iint_{\Omega\times\Omega} \frac{\mathrm{d}x\,\mathrm{d}y}{|x-y|^\lambda}\,.$$

The perimeter $\operatorname{Per}\Omega$ is taken in the sense of De Giorgi, namely

$$\operatorname{Per}\ \Omega = \sup\left\{\int_\Omega \operatorname{div}F(x)\,\mathrm{d}x \mid F \in C_0^1(\mathbb{R}^3, \mathbb{R}^3), |F| \leq 1\right\},$$

which is simply the surface area of Ω when the boundary is smooth. We consider the minimization problem

$$E(m) = \inf_{|\Omega|=m} \mathcal{E}(\Omega).$$

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The most important case is $\lambda = 1$ in dimension $N = 3$, which goes back to Gamow's liquid drop model for atomic nuclei [15]. In this case, a nucleus is thought of consisting of nucleons (protons and neutrons) in a set $\Omega \subset \mathbb{R}^N$. The nucleons are assumed to be concentrated with constant density, which implies that the number of nucleons is proportional to $|\Omega|$. The perimeter term in the energy functional corresponds to a surface tension, which holds the nuclei together. The second term in the energy functional corresponds to a Coulomb repulsion among the protons. Here for simplicity we have scaled all physical constants to be unity.

In the last decade, this model (for general λ and N) has gained renewed interest in the mathematics literature. We refer to [6] for a review and, for instance, to [2,9–12,16–19,22] and references therein; see also [14,25]. A variant of the problem with a constant background has also been intensely studied, see, for instance, [1,3–5,8,13,20] and references therein.

In principle, the two terms in $\mathcal{E}(\Omega)$ are competing against each other: balls minimize the first term (by the isoperimetric inequality [7], see also [23, Theorem 14.1]) and maximize the second term (by the Riesz rearrangement inequality [26], see also [21, Theorem 3.7]). Thus the question about the existence of a minimizer for $E(m)$ is nontrivial.

Clearly, the existence will depend on the parameter $m > 0$. By scaling $\Omega \mapsto m^{1/N} U$ with $|U| = 1$, we see that

$$\mathcal{E}(\Omega) = m^{\frac{N-1}{N}} \operatorname{Per} U + m^{\frac{2N-\lambda}{N}} D(U) = m^{\frac{N-1}{N}} \Big(\operatorname{Per} U + m^{\frac{N+1-\lambda}{N}} D(U) \Big).$$

Note that $(N + 1 - \lambda)/N > 0$. This suggests that for small m the short range attraction due to the perimeter term is dominant, whereas for large m the long range repulsion due to the Riesz potential is dominant. Correspondingly, we expect that there is a minimizer for small m and there is no minimizer for large m.

In the case $\lambda = 1$, $N = 3$, the physics literature suggests that there is a critical volume $m_* > 0$ such that balls are unique minimizers for $E(m)$ when $m \leq m_*$ and there is no minimizer when $m > m_*$. The value m_* corresponds to the threshold where the energy of a ball of volume m is equal to that of two balls of mass $m/2$, spaced infinitely far apart. It can be computed explicitly to be (see [5,12])

$$m_* = \frac{|B_1| \ \operatorname{Per} B_1}{D(B_1)} \cdot \frac{2^{1/3} - 1}{1 - 2^{-2/3}} = 5 \frac{2^{1/3} - 1}{1 - 2^{-2/3}} \approx 3.512$$

with B_1 the unit ball in $\mathbb{R}^3$. A mathematical proof of this remains unknown.

In the present paper, we consider the general case $N \geq 2$ and $\lambda \in (0, N)$. We define the critical volume m_* to be the unique value such that

$$\mathcal{E}\Big(\Big(\frac{m_*}{|B_1|}\Big)^{\frac{1}{N}} B_1\Big) = 2\, \mathcal{E}\Big(\Big(\frac{m_*}{2|B_1|}\Big)^{1/N} B_1\Big),$$

namely,

$$m_* = \Big(\frac{2^{1/N} - 1}{1 - 2^{(\lambda - N)/N}} \cdot \frac{\operatorname{Per} B_1}{D(B_1)} \Big)^{N/(N+1-\lambda)} |B_1|. \tag{1}$$

Here B_1 is the unit ball in $\mathbb{R}^N$ (hence, $(m/|B_1|)^{1/N} B_1$ is a ball of measure m). Thus, just like in the special case $\lambda = 1$, $N = 3$, this is the critical value where the energy of a ball of volume m_* is equal to that of two balls of mass $m_*/2$, spaced infinitely far apart, and it is natural to conjecture that m_* divides the regime where minimizers are balls from the regime where there are no minimizers.

The following results were proved by Knüpfer and Muratov [18,19]:

(a) For every $N \geq 2$ and $\lambda \in (0, N)$, there exists a constant $m_{c_1} > 0$ such that $E(m)$ has a minimizer for every $m \leq m_{c_1}$.
(b) For every $N \geq 2$ and $\lambda \in (0, 2)$, there exists a constant $m_{c_2} > 0$ such that $E(m)$ has no minimizer for every $m > m_{c_2}$.
(c) If $N = 2$ and $\lambda > 0$ is sufficiently small, then $m_{c_1} = m_{c_2} = m_*$ and balls are unique minimizers for $E(m)$ with $m \leq m_*$.
(d) if $N = 2$ and $\lambda < 2$, or if $3 \leq N \leq 7$ and $\lambda < N - 1$, then there exists a constant $0 < m'_{c_1} \leq m_{c_1}$ such that balls are unique minimizers for $E(m)$ with $m < m'_{c_1}$.

In the most important case $\lambda = 1$, $N = 3$, see also [11,22] for alternative proofs of the non-existence result (b) and [16] for a short proof of the uniqueness result (d). In [24], Muratov and Zaleski proved (c) for the explicit range $0 < \lambda \leq 0.034$ and $N = 2$. In [2], Bonacini and Cristoferi extended (c) and (d) to all $N \geq 2$. In [9], Figalli, Fusco, Maggi, Millot and Morini extended (d) to all $N \geq 2$ and $\lambda \in (0, N)$.

Our first new result concerns the existence in (a). Except when $\lambda > 0$ is small, the existence of minimizers for $E(m)$ is known only for small m. In this paper, we extend the existence to what is conjectured to be the optimal range of parameters.

Theorem 1 *Let $N \geq 2$ and $\lambda \in (0, N)$. Then the variational problem $E(m)$ has a minimizer for every $0 < m \leq m_*$, where m_* is defined in (1).*

We will prove Theorem 1 by establishing the strict binding inequality [12]

$$E(m) < E(m_1) + E(m - m_1), \quad \forall 0 < m_1 < m \tag{2}$$

for all $m < m_*$. As a by product of our proof, we obtain the following conditional uniqueness of minimizers.

Theorem 2 *Let $N \geq 2$ and $\lambda \in (0, N)$. If $E(m)$ has no minimizer when $m > m_*$, then balls are minimizers for $E(m)$ when $m \leq m_*$ and they are unique minimizers when $m < m_*$.*

So far, the non-existence result in the sharp range $m > m_*$ is only available for $\lambda > 0$ small [2,18,24]. For larger λ and a nonexplicit range of m, we have

Theorem 3 *Let $N \geq 2$ and $\lambda \in (0, N)$ and $\lambda \leq 2$. Then there exists a constant $m_{c_2} \geq m_*$ such that $E(m)$ does not have a minimizer for all $m > m_{c_2}$.*

This result is due to [18,19,22] for $\lambda < 2$ and seems to be unpublished for $\lambda = 2$. We will combine the methods in [11] and [18,19]. It is an open problem whether the nonexistence result also holds for $2 < \lambda < N$ when $N \geq 3$.

2 Existence

In this section we prove Theorem 1. We will deduce Theorem 1 from the following strict binding inequality.

Theorem 4 *Let $N \geq 2$ and $\lambda \in (0, N)$. Then for every $0 < m < m_*$ with m_* in (1), we have*

$$E(m) < E(m_1) + E(m - m_1), \quad \forall 0 < m_1 < m. \tag{3}$$

Thanks to [12, Theorem 3.1], the strict binding inequality (3) is a sufficient condition for the existence of minimizers of $E(m)$. Moreover, by [12, Theorem 3.4], the set $\{m > 0 : E(m) \text{ has a minimizer}\}$ is closed in $(0, \infty)$. Hence, Theorem 4 implies the existence of minimizers of $E(m)$ for all $0 < m \le m_*$. Note that the proofs of Theorems 3.1 and 3.4 in [12] extend, without modifications, to the case $\lambda \neq 1$; see Remark 3.7 in that paper.

We will prove the strict binding inequality using a scaling argument, based on the following key observation which uses only the isoperimetric inequality.

Lemma 5 *If $0 < m_1 < m$, then we have, with $s = m_1/m \in (0,1)$ and B_1 the unit ball in $\mathbb{R}^N$,*

$$E(m_1) \ge s^{(2N-\lambda)/N} E(m) + (1 - s^{(N+1-\lambda)/N}) s^{(N-1)/N} \left(\frac{m}{|B_1|}\right)^{(N-1)/N} \text{Per } B_1.$$

Proof Take $\Omega \subset \mathbb{R}^N$ such that $|\Omega| = m_1$. Then $|s^{-1/N}\Omega| = m$, and hence

$$\begin{aligned} E(m) &\le \mathcal{E}(s^{-1/N}\Omega) = s^{-(N-1)/N} \text{Per } \Omega + s^{-(2N-\lambda)/N} D(\Omega) \\ &= s^{-(2N-\lambda)/N} \mathcal{E}(\Omega) - \left(s^{-(2N-\lambda)/N} - s^{-(N-1)/N}\right) \text{Per } \Omega. \end{aligned}$$

By the isoperimetric inequality

$$\text{Per } \Omega \ge \left(\frac{m_1}{|B_1|}\right)^{(N-1)/N} \text{Per } B_1 = s^{(N-1)/N} \left(\frac{m}{|B_1|}\right)^{(N-1)/N} \text{Per } B_1.$$

Thus

$$E(m) \le s^{-(2N-\lambda)/N} \mathcal{E}(\Omega) - \left(s^{-(2N-\lambda)/N} - s^{-(N-1)/N}\right) s^{(N-1)/N} \left(\frac{m}{|B_1|}\right)^{(N-1)/N} \text{Per } B_1.$$

Optimizing over all Ω satisfying $|\Omega| = m_1$ we get

$$E(m) \le s^{-(2N-\lambda)/N} E(m_1) - \left(s^{-(2N-\lambda)/N} - s^{-(N-1)/N}\right) s^{(N-1)/N} \left(\frac{m}{|B_1|}\right)^{(N-1)/N} \text{Per } B_1$$

which is equivalent to the desired inequality. □

Proof of Theorem 4 Take $0 < m_1 < m < m_*$. Denote $s = m_1/m \in (0,1)$. By Lemma 5 we have

$$\begin{aligned} E(m_1) &\ge s^{(2N-\lambda)/N} E(m) + (1 - s^{(N+1-\lambda)/N}) s^{(N-1)/N} \left(\frac{m}{|B_1|}\right)^{(N-1)/N} \text{Per } B_1, \\ E(m - m_1) &\ge (1-s)^{(2N-\lambda)/N} E(m) \\ &\quad + (1 - (1-s)^{(N+1-\lambda)/N})(1-s)^{(N-1)/N} \left(\frac{m}{|B_1|}\right)^{(N-1)/N} \text{Per } B_1. \end{aligned}$$

Therefore,

$$\begin{aligned} E(m_1) + E(m - m_1) - E(m) &\ge \left(s^{(2N-\lambda)/N} + (1-s)^{(2N-\lambda)/N} - 1\right) E(m) \\ &+ \left((1 - s^{(N+1-\lambda)/N}) s^{(N-1)/N} + (1 - (1-s)^{(N+1-\lambda)/N})(1-s)^{(N-1)/N}\right) \\ &\times \left(\frac{m}{|B_1|}\right)^{(N-1)/N} \text{Per } B_1. \end{aligned} \tag{4}$$

Moreover, by the variational principle,

$$E(m) \le \mathcal{E}\left(\left(\frac{m}{|B_1|}\right)^{1/N} B_1\right) = \left(\frac{m}{|B|}\right)^{(N-1)/N} \text{Per } B_1 + \left(\frac{m}{|B_1|}\right)^{(2N-\lambda)/N} D(B_1). \tag{5}$$

Inserting (5) in (4) and using

$$s^{(2N-\lambda)/N} + (1-s)^{(2N-\lambda)/N} - 1 < 0, \quad \forall s \in (0,1), \tag{6}$$

we find that

$$\begin{aligned}
&E(m_1) + E(m-m_1) - E(m) \\
&\geq \Big(s^{(2N-\lambda)/N} + (1-s)^{(2N-\lambda)/N} - 1\Big)\Big(\Big(\frac{m}{|B|}\Big)^{(N-1)/N} \operatorname{Per} B_1 + \Big(\frac{m}{|B_1|}\Big)^{(2N-\lambda)/N} D(B_1)\Big) \\
&\quad + \Big((1 - s^{(N+1-\lambda)/N})s^{(N-1)/N} + (1-(1-s)^{(N+1-\lambda)/N})(1-s)^{(N-1)/N}\Big) \\
&\quad \times \Big(\frac{m}{|B_1|}\Big)^{(N-1)/N} \operatorname{Per}\ B_1 \\
&= \Big(s^{(N-1)/N} + (1-s)^{(N-1)/N} - 1\Big)\Big(\frac{m}{|B_1|}\Big)^{(N-1)/N} \operatorname{Per}\ B_1 \\
&\quad + \Big(s^{(2N-\lambda)/N} + (1-s)^{(2N-\lambda)/N} - 1\Big)\Big(\frac{m}{|B_1|}\Big)^{(2N-\lambda)/N} D(B_1) \\
&= \Big(s^{(2N-\lambda)/N} + (1-s)^{(2N-\lambda)/N} - 1\Big)\Big(\frac{m}{|B_1|}\Big)^{(N-1)/N} \operatorname{Per}\ B_1 \\
&\quad \times \Big(\frac{D(B_1)}{\operatorname{Per}\ B_1}\Big(\frac{m}{|B_1|}\Big)^{(N+1-\lambda)/N} - f(s)\Big)
\end{aligned}$$

with

$$f(s) := \frac{s^{(N-1)/N} + (1-s)^{(N-1)/N} - 1}{1 - s^{(2N-\lambda)/N} - (1-s)^{(2N-\lambda)/N}}. \tag{7}$$

Using again (6), we find that the strict binding inequality

$$E(m_1) + E(m-m_1) - E(m) > 0$$

holds true if

$$f(s) > \frac{D(B_1)}{\operatorname{Per}\ B_1}\Big(\frac{m}{|B_1|}\Big)^{(N+1-\lambda)/N}, \quad \forall s \in (0,1). \tag{8}$$

On the other hand, we can show that (see Lemma 6 below)

$$\min_{s\in(0,1)} f(s) = f(1/2) = \frac{2^{1/N} - 1}{1 - 2^{(\lambda-N)/N}}. \tag{9}$$

Therefore, (8) holds true when

$$\frac{2^{1/N} - 1}{1 - 2^{(\lambda-N)/N}} > \frac{D(B_1)}{\operatorname{Per}\ B_1}\Big(\frac{m}{|B_1|}\Big)^{(N+1-\lambda)/N}$$

which is equivalent to $m < m_*$. □

It remains to prove (9). We have

Lemma 6 *For all* $0 < a < 1 < b < 2$ *and* $0 < s < 1$ *we have*

$$\frac{s^a + (1-s)^a - 1}{2^{1-a} - 1} \geq -\frac{s\log s + (1-s)\log(1-s)}{\log 2} \geq \frac{s^b + (1-s)^b - 1}{2^{1-b} - 1}.$$

The inequality (9) follows from Lemma 6 with $a = (N-1)/N$ and $b = (2N-\lambda)/N$.

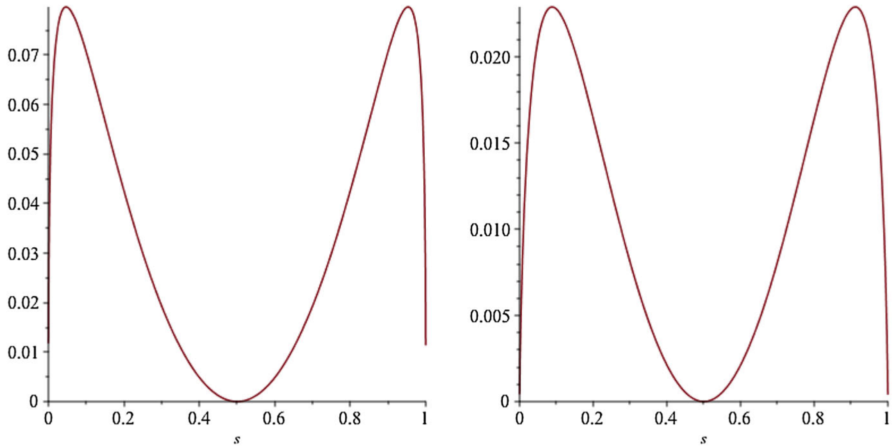

Fig. 1 The function $g(s)$, $s \in (0, 1)$, with $\alpha = 0.5$ (left) and $\alpha = 1.5$ (right)

Proof We will prove that for all $\alpha \in (0, 2)$ and all $s \in (0, 1)$,

$$g(s) := s^\alpha + (1-s)^\alpha - 1 + \frac{2^{1-\alpha}-1}{\log 2}\Big(s\log s + (1-s)\log(1-s)\Big) \geq 0. \tag{10}$$

Then the desired conclusion follows from (10) and the fact that $2^{1-\alpha} - 1 > 0$ if $\alpha \in (0, 1)$ while $2^{1-\alpha} - 1 < 0$ if $\alpha \in (1, 2)$ (Fig. 1).

By the symmetry $s \leftrightarrow 1-s$, it suffices to prove (10) for $s \in (0, 1/2]$. Also, (10) is trivial when $\alpha = 1$, so we will distinguish two cases $\alpha \in (0, 1)$ and $\alpha \in (1, 2)$.

Case 1: $\alpha \in (0, 1)$. We have

$$g'(s) = \alpha\Big(s^{\alpha-1} - (1-s)^{\alpha-1}\Big) + \frac{2^{1-\alpha}-1}{\log 2}\Big(\log s - \log(1-s)\Big),$$

$$g''(s) = \alpha(\alpha-1)\Big(s^{\alpha-2} + (1-s)^{\alpha-2}\Big) + \frac{2^{1-\alpha}-1}{\log 2}\Big(s^{-1} + (1-s)^{-1}\Big).$$

Define $h : (0, 1/2] \to \mathbb{R}$ by

$$\begin{aligned} h(s) &:= s(1-s)g''(s) \\ &= \alpha(\alpha-1)s(1-s)\Big(s^{\alpha-2} + (1-s)^{\alpha-2}\Big) + \frac{2^{1-\alpha}-1}{\log 2} \\ &= \alpha(\alpha-1)\Big(s^{\alpha-1} + (1-s)^{\alpha-1} - s^\alpha - (1-s)^\alpha\Big) + \frac{2^{1-\alpha}-1}{\log 2}. \end{aligned}$$

Note that for all $s \in (0, 1/2)$ we have

$$h'(s) = \alpha(\alpha-1)^2\Big(s^{\alpha-2} - (1-s)^{\alpha-2}\Big) + \alpha^2(1-\alpha)\Big(s^{\alpha-1} - (1-s)^{\alpha-1}\Big) > 0$$

since

$$s^{\alpha-2} - (1-s)^{\alpha-2} > 0, \quad (1-\alpha)(s^{\alpha-1} - (1-s)^{\alpha-1}) > 0.$$

Thus h is strictly increasing on $(0, 1/2]$. Moreover,

$$\lim_{s\to 0^+} h(s) = -\infty$$

and

$$h(1/2) = \alpha(\alpha-1)2^{1-\alpha} + \frac{2^{1-\alpha}-1}{\log 2} = 2^{1-\alpha}\Big(\frac{1}{\log 2} - \alpha(1-\alpha)\Big) - \frac{1}{\log 2}$$
$$\geq \Big(1 + (1-\alpha)\log 2\Big)\Big(\frac{1}{\log 2} - \alpha(1-\alpha)\Big) - \frac{1}{\log 2}$$
$$= (1-\alpha)^2\Big[1 - \alpha\log 2\Big] > 0,$$

since

$$2^{1-\alpha} = e^{(1-\alpha)\log 2} \geq 1 + (1-\alpha)\log 2, \quad \alpha(1-\alpha) \leq \frac{1}{4} < \frac{1}{\log 2}.$$

Thus there exists a unique value $s_1 \in (0, 1/2)$ (depending on α) such that

$$h(s) < 0 \text{ on } s \in (0, s_1), \quad h(s) > 0 \text{ on } s \in (s_1, 1/2).$$

Putting back the definition $h(s) = s(1-s)g''(s)$, we find that

$$g''(s) < 0 \text{ on } s \in (0, s_1), \quad g''(s) > 0 \text{ on } s \in (s_1, 1/2).$$

Thus $g'(s)$ is strictly decreasing on $s \in (0, s_1)$ and strictly increasing on $s \in (s_1, 1/2)$. Combining with

$$\lim_{s\to 0^+} g'(s) = \infty, \quad g'(1/2) = 0,$$

we find that there exists a unique value $s_2 \in (0, 1/2)$ (depending on α) such that

$$g'(s) > 0 \text{ on } s \in (0, s_2), \quad g'(s) < 0 \text{ on } s \in (s_2, 1/2).$$

Thus $g(s)$ is strictly increasing on $s \in (0, s_2)$ and strictly decreasing on $s \in (s_2, 1/2)$. Therefore,

$$\inf_{s\in(0,1/2]} g(s) = \min\{\lim_{s\to 0^+} g(s), g(1/2)\} = 0.$$

Case 2: $\alpha \in (1, 2)$. We can proceed similarly. To be precise, the function $h(s) = s(1-s)g''(s)$ also satisfies

$$h'(s) = \alpha(\alpha-1)^2\Big(s^{\alpha-2} - (1-s)^{\alpha-2}\Big) + \alpha^2(1-\alpha)\Big(s^{\alpha-1} - (1-s)^{\alpha-1}\Big) > 0$$

for all $s \in (0, 1/2)$. Thus h is also strictly increasing on $(0, 1/2]$. Moreover,

$$\lim_{s\to 0^+} h(s) = \frac{2^{1-\alpha}-1}{\log 2} < 0$$

and

$$h(1/2) = \alpha(\alpha-1)2^{1-\alpha} + \frac{2^{1-\alpha}-1}{\log 2} = \alpha(\alpha-1)2^{1-\alpha} + \frac{e^{(1-\alpha)\log 2}-1}{\log 2}$$
$$\geq \alpha(\alpha-1)2^{1-\alpha} + 1 - \alpha = (\alpha-1)(\alpha 2^{1-\alpha} - 1) > 0.$$

Here we have used $\alpha 2^{1-\alpha} > 1$ for all $\alpha \in (1, 2)$ (the function $q(\alpha) = \alpha 2^{1-\alpha}$ is concave on $(1, 2)$ as $q''(\alpha) = 2^{1-\alpha}(\log 2)(\alpha \log 2 - 2) < 0$ and $q(1) = q(2) = 1$).

Thus there exists a unique value $s_1 \in (0, 1/2)$ (depending on α) such that

$$h(s) < 0 \text{ on } s \in (0, s_1), \quad h(s) > 0 \text{ on } s \in (s_1, 1/2).$$

The rest is exactly the same as in Case 1. This completes the proof of Lemma 6. □

3 Uniqueness

Proof of Theorem 2 *Step 1* We prove that balls are minimizers for $E(m_*)$. Assume by contradiction that balls are not minimizers for $E(m_*)$, namely

$$E(m_*) < \mathcal{E}((m_*/|B_1|)^{1/N} B_1).$$

Since $m \mapsto E(m)$ and $m \mapsto \mathcal{E}((m/|B_1|)^{1/N} B_1)$ are continuous (for the first function, see [12, Proof of Theorem 3.1]), there exists a constant $\delta \in (0, 1)$ such that for all $m \in [m_*, m_* + \delta)$ we have

$$\begin{aligned} E(m) &\le (1-\delta)\mathcal{E}((m/|B_1|)^{1/N} B_1) \\ &\le \left(\frac{m}{|B_1|}\right)^{(N-1)/N} \text{Per } B_1 + \left(\frac{m}{|B_1|}\right)^{(2N-\lambda)/N} (1-\delta) D(B_1). \end{aligned} \tag{11}$$

This is similar to (5), but $D(B_1)$ is replaced by $(1-\delta)D(B_1)$. Proceeding similarly as in the proof of Theorem 4 and inserting (11) (instead of (5)) in (4), for all $m \in [m_*, m_* + \delta)$ and $0 < m_1 < m$ we have

$$\begin{aligned} &E(m_1) + E(m - m_1) - E(m) \\ &\ge \left(s^{(2N-\lambda)/N} + (1-s)^{(2N-\lambda)/N} - 1\right)\left(\frac{m}{|B_1|}\right)^{(N-1)/N} \text{Per } B_1 \times \\ &\times \left(\frac{(1-\delta)D(B_1)}{\text{Per } B_1}\left(\frac{m}{|B_1|}\right)^{(N+1-\lambda)/N} - f(s)\right) \end{aligned}$$

with $s = m_1/m \in (0, 1)$ and with the same function $f(s)$ in (7). By (9), we conclude that

$$E(m_1) + E(m - m_1) - E(m) > 0, \quad \forall 0 < m_1 < m$$

provided that

$$\frac{2^{1/N} - 1}{1 - 2^{(\lambda-N)/N}} \ge \frac{(1-\delta)D(B_1)}{\text{Per } B_1}\left(\frac{m}{|B_1|}\right)^{(N+1-\lambda)/N}$$

which is equivalent to

$$m \le m_*(1-\delta)^{-N/(N+1-\lambda)}.$$

Thus the variational problem $E(m)$ has a minimizer for all

$$m \le \min\{m_* + \delta, m_*(1-\delta)^{-N/(N+1-\lambda)}\}.$$

This is a contradiction to the assumption that $E(m)$ has no minimizer if $m > m_*$. Thus we conclude that balls are minimizers for $E(m_*)$.

Step 2 Now we prove that if $m < m_*$, then balls are unique minimizers for $E(m)$. This fact follows from [2, Theorem 2.10] which states that the set where balls are minimizers is an interval and that one has uniqueness away from the endpoint (note that this part does not require the assumption $\lambda < N - 1$ which is imposed in the rest of [2]). For the reader's convenience, we provide a direct proof below.

Consider an arbitrary measurable set $\Omega \subset \mathbb{R}^N$ with $|\Omega| = m < m_*$. Then proceeding as in the proof of Lemma 5, we find that

$$\mathcal{E}(\Omega) \ge s^{(2N-\lambda)/N} E(m_*) + (1 - s^{(N+1-\lambda)/N}) s^{(N-1)/N} \left(\frac{m_*}{|B_1|}\right)^{(N-1)/N} \text{Per } B_1 \tag{12}$$

with $s = m/m_* \in (0,1)$ and the equality occurs if and only if Ω is a ball. On the other hand, we know that balls are minimizers for $E(m_*)$, namely

$$E(m_*) = \mathcal{E}\Big(\Big(\frac{m_*}{|B_1|}\Big)^{1/N} B_1\Big) = \Big(\frac{m_*}{|B_1|}\Big)^{(N-1)/N} \operatorname{Per} B_1 + \Big(\frac{m_*}{|B_1|}\Big)^{(2N-\lambda)/N} D(B_1).$$

Inserting the latter equality in (12), we obtain

$$\begin{aligned}\mathcal{E}(\Omega) &\geq s^{(2N-\lambda)/N}\Big(\Big(\frac{m_*}{|B|}\Big)^{(N-1)/N} \operatorname{Per} B_1 + \Big(\frac{m_*}{|B_1|}\Big)^{(2N-\lambda)/N} D(B_1)\Big) \\ &\quad + (1 - s^{(N+1-\lambda)/N}) s^{(N-1)/N} \Big(\frac{m_*}{|B_1|}\Big)^{(N-1)/N} \operatorname{Per} B_1 \\ &= \Big(\frac{m}{|B_1|}\Big)^{(N-1)/N} \operatorname{Per} B_1 + \Big(\frac{m}{|B_1|}\Big)^{(2N-\lambda)/N} D(B_1) = \mathcal{E}\Big(\Big(\frac{m}{B_1}\Big)^{1/N} B_1\Big).\end{aligned}$$

Thus balls are minimizers for $E(m)$; moreover, if Ω is a minimizer for $E(m)$, then the equality occurs in (12) and Ω is a ball. □

4 Nonexistence

In this section we prove Theorem 3. First, by extending the analysis for $\lambda = 1$ in [11] to general λ, we have

Lemma 7 *Let $N \geq 2$ and $\lambda \in (0, N)$. Let $m > 0$ be arbitrary. Let $\Omega \subset \mathbb{R}^N$ be a minimizer for $E(m)$. Then*

$$\iint_{\Omega\times\Omega} \frac{dx\,dy}{|x-y|^{\lambda-1}} \lesssim |\Omega|\,,$$

with an implied constant depending only on λ and N.

It is unclear to us whether the power $\lambda - 1$ in Lemma 7 is optimal. If we could replace this power by a smaller one, then we would be able to improve the condition $\lambda \leq 2$ in Theorem 3.

Proof For $\nu \in \mathbb{S}^{N-1}$ and $t \in \mathbb{R}$ we set

$$\Omega^{\pm}_{\nu,t} := \Omega \cap \{x \in \mathbb{R}^N : \pm\nu \cdot x > \pm t\}\,.$$

For any $\rho \geq 0$, the set

$$\Omega^+_{\nu,t} \cup \big(\Omega^-_{\nu,t} - \rho\nu\big)$$

has measure $|\Omega^+_{\nu,t} \cup \big(\Omega^-_{\nu,t} - \rho\nu\big)| = |\Omega| = m$ and therefore, by minimality of Ω,

$$\mathcal{E}\big(\Omega^+_{\nu,t} \cup \big(\Omega^-_{\nu,t} - \rho\nu\big)\big) \geq \mathcal{E}(\Omega)\,. \tag{13}$$

For any $\rho > 0$, we have

$$\operatorname{Per}\big(\Omega^+_{\nu,t} \cup \big(\Omega^-_{\nu,t} - \rho\nu\big)\big) = \operatorname{Per}\Omega^+_{\nu,t} + \operatorname{Per}\Omega^-_{\nu,t} \leq \operatorname{Per}\Omega + 2\sigma(\Omega \cap \{\nu\cdot x = t\})\,,$$

where σ denotes the induced measure on the hyperplane $\{\nu \cdot x = t\}$ and where the inequality holds for almost every $t \in \mathbb{R}$.

On the other hand, for any $\rho \geq 0$,

$$\iint_{(\Omega^+_{\nu,t}\cup(\Omega^-_{\nu,t}-\rho\nu))\times(\Omega^+_{\nu,t}\cup(\Omega^-_{\nu,t}-\rho\nu))} \frac{dx\,dy}{|x-y|^{\lambda}} = \iint_{\Omega^+_{\nu,t}\times\Omega^+_{\nu,t}} \frac{dx\,dy}{|x-y|^{\lambda}} + \iint_{\Omega^-_{\nu,t}\times\Omega^-_{\nu,t}} \frac{dx\,dy}{|x-y|^{\lambda}}$$

$$+2\iint_{\Omega^+_{\nu,t}\times\Omega^-_{\nu,t}} \frac{dx\,dy}{|x-y+\rho\nu|^\lambda}\,.$$

The last double integral tends to zero as $\rho\to\infty$

Inserting these facts into (13) and letting $\rho\to\infty$, we infer

$$\begin{aligned}
&\operatorname{Per}\Omega + 2\sigma(\Omega\cap\{\nu\cdot x=t\}) + \frac12\iint_{\Omega^+_{\nu,t}\times\Omega^+_{\nu,t}}\frac{dx\,dy}{|x-y|^\lambda} + \frac12\iint_{\Omega^-_{\nu,t}\times\Omega^-_{\nu,t}}\frac{dx\,dy}{|x-y|^\lambda}\\
&\quad\geq \mathcal{E}(\Omega)\\
&\quad= \operatorname{Per}\Omega + \frac12\iint_{\Omega^+_{\nu,t}\times\Omega^+_{\nu,t}}\frac{dx\,dy}{|x-y|^\lambda} + \frac12\iint_{\Omega^-_{\nu,t}\times\Omega^-_{\nu,t}}\frac{dx\,dy}{|x-y|^\lambda} + \iint_{\Omega^+_{\nu,t}\times\Omega^-_{\nu,t}}\frac{dx\,dy}{|x-y|^\lambda}\,,
\end{aligned}$$

that is,

$$\sigma(\Omega\cap\{\nu\cdot x=t\}) \geq \frac12\iint_{\Omega^+_{\nu,t}\times\Omega^-_{\nu,t}}\frac{dx\,dy}{|x-y|^\lambda}\,.$$

Note that the double integral here can be written as $\iint_{\Omega\times\Omega}|x-y|^{-\lambda}\mathbb{1}_{\{\nu\cdot x>t>\nu\cdot y\}}\,dx\,dy$. Thus, integrating the inequality with respect to $t\in\mathbb{R}$ gives, by Fubini's theorem,

$$|\Omega| \geq \frac12\iint_{\Omega\times\Omega}\frac{(\nu\cdot(x-y))_+}{|x-y|^\lambda}\,dx\,dy\,.$$

Finally, we average this inequality with respect to $\nu\in\mathbb{S}^{N-1}$ and use the fact that

$$\int_{\mathbb{S}^{N-1}}(\nu\cdot(x-y))_+\frac{d\nu}{|\mathbb{S}^{N-1}|} = c_N|x-y|\,,$$

to obtain the bound in the lemma. □

With Lemma 7 at hand, it is easy to finish the proof of Theorem 3 if $\lambda\leq 1$. In fact, if $\lambda=1$, the lemma gives directly $|\Omega|^2\lesssim|\Omega|$, which is the claimed bound. If $0<\lambda<1$, by Riesz's rearrangement inequality we have for all $t>0$

$$\iint_{\Omega\times\Omega} t^{-1}(1-e^{-t|x-y|^{1-\lambda}})\,dx\,dy \geq \iint_{\Omega^*\times\Omega^*} t^{-1}(1-e^{-t|x-y|^{1-\lambda}})\,dx\,dy\,,$$

where Ω^* is the ball centered at 0 with volume $|\Omega^*|=|\Omega|$. Taking $t\to 0$ we obtain

$$\iint_{\Omega\times\Omega}\frac{dx\,dy}{|x-y|^{\lambda-1}} \geq \iint_{\Omega^*\times\Omega^*}\frac{dx\,dy}{|x-y|^{\lambda-1}} \gtrsim |\Omega^*|^{(2N-\lambda+1)/N} = |\Omega|^{(2N-\lambda+1)/N}\,.$$

Since $(2N-\lambda+1)/N>1$, the lemma implies once again $|\Omega|\lesssim 1$.

It remains to deal with the case $1<\lambda\leq 2$. The key is the following bound, which in the special case $N=3$ and $\lambda=1$ appears in [22, Eq. (2.12)]. The proof there extends immediately to the general case, since the analogues of [22, Lemma 3 (ii) and Lemma 4] hold according to [19, Lemmas 4.1 and 4.3].

Lemma 8 *Let $N\geq 2$ and $\lambda\in(0,N)$. Let $m\geq\omega_N$ and let $\Omega\subset\mathbb{R}^N$ be a minimizer for $E(m)$. Then, for $1\leq R\leq\operatorname{diam}\Omega$,*

$$|\Omega\cap B_R(x)|\gtrsim R \qquad \text{for a.e. } x\in\Omega\,,$$

with an implied constant depending only on λ and N.

Here diam Ω in the lemma is understood as the diameter of the set $\{x \in \mathbb{R}^N : |\Omega \cap B_r(x)| > 0 \text{ for all } r > 0\}$.

We will use this lemma to deduce Theorem 3 for $1 < \lambda \leq 2$. If diam $\Omega \leq 2$, then $|\Omega| \lesssim 1$ and we are done. Thus, assuming diam $\Omega > 2$, we have, by Lemma 8,

$$\begin{aligned}
\iint_{\Omega\times\Omega} \frac{dx\,dy}{|x-y|^{\lambda-1}} &= (\lambda-1)\int_0^\infty \frac{dR}{R^\lambda}\,|\{(x,y)\in\Omega\times\Omega:\ |x-y|<R\}| \\
&= (\lambda-1)\int_0^\infty \frac{dR}{R^\lambda}\int_\Omega dx\,|\Omega\cap B_R(x)| \\
&\geq (\lambda-1)\int_1^{\operatorname{diam}\Omega} \frac{dR}{R^\lambda}\int_\Omega dx\,|\Omega\cap B_R(x)| \\
&\gtrsim \int_1^{\operatorname{diam}\Omega} \frac{dR}{R^{\lambda-1}}\,|\Omega|\,.
\end{aligned}$$

The right side is bounded from below by a constant times $|\Omega|(\operatorname{diam}\Omega)^{2-\lambda}$ if $\lambda < 2$ and by a constant times $|\Omega| \log \operatorname{diam}\Omega$ if $\lambda = 2$. Combining this lower bound on the double integral with the upper bound from Lemma 7, we infer in either case that diam $\Omega \lesssim 1$, which implies $|\Omega| \lesssim 1$ and therefore concludes the proof of Theorem 3.

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