Testification of Condorcet Winners in Dueling Bandits
Supplemental Material

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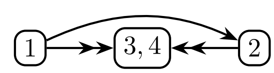
LIST OF SYMBOLS

$\mathbb{N}$	set of natural numbers (without 0), i.e., $\mathbb{N} = \{1, 2, 3, \dots\}$
$\mathbb{N}_0$	set of natural numbers including zero, i.e., $\mathbb{N}_0 = \{0, 1, 2, 3, \dots\}$
$[m]$	the set $\{1, 2, \dots, m\}$
$[m]_2$	the set of all subsets of size 2 of $[m]$
$(m)_2$	the set $\{(i, j) \in [m] \times [m] \mid i < j\}$
$\langle m \rangle_2$	the set $\{(i, j) \in [m] \times [m] \mid i \neq j\}$
$\mathbf{1}_A$	indicator function on the set A
$\mathrm{Ber}(p)$	Bernoulli distribution with success probability $p \in [0, 1]$
$\mathcal{A}$	an algorithm
$\mathcal{A}[i]$	the algorithm $\mathcal{A}$ with input i
$\mathbf{D}(\mathcal{A})$	the return value/ decision of algorithm $\mathcal{A}$
$T^{\mathcal{A}}$	termination time of $\mathcal{A}$
$\boldsymbol{n}_0, \boldsymbol{w}_0$	initialization parameters (assumed to be $\boldsymbol{n}_0 = \boldsymbol{w}_0 = (0)_{1 \le i,j \le m}$)
$\mathbf{Q}$	relation $(q_{i,j})_{1 \le i,j \le m} \in \mathcal{Q}_m$, where $q_{i,j} = \mathbb{P}(i \succ j)$ is the (unknown) ground truth pairwise winning probability that i wins when compared to j
$\mathcal{Q}_m$	set of reciprocal relations on $[m]$
$\mathcal{Q}_m(\mathrm{CW})$	set of all those $\mathbf{Q} \in \mathcal{Q}_m$, which have a CW.
$\mathcal{Q}_m(i^*)$	set of all those $\mathbf{Q} \in \mathcal{Q}_m$, which have i^* as CW.
$\mathcal{Q}_m(\neg X)$	$\mathcal{Q}_m \setminus \mathcal{Q}_m(X)$; here, $X \in \{\mathrm{CW}, i^*\}$.
$\mathcal{Q}_m^h$	set of all $\mathbf{Q} = (q_{i,j})_{1 \le i,j \le m} \in \mathcal{Q}_m$ with $\lvert q_{i,j} - 1/2 \rvert > h$ for all $i \neq j \in [m]$
$\mathcal{Q}_m^h(X)$	$\mathcal{Q}_m^h \cap \mathcal{Q}_m(X)$; here, $X \in \{\mathrm{CW}, \neg\mathrm{CW}, i^*, \neg i^*\}$
$\mathcal{P}^{m,h,\alpha,\beta}$	short for: testification for the CW on $\mathcal{Q}_m^h$ for α and β
$X_{i,j}^{[t]}$	the outcome of the comparison between i and j at time t, $\sim \mathrm{Ber}(q_{i,j})$
$(\boldsymbol{w}_t)_{i,j}$	the number of wins of arm i against arm j until time t
$(\boldsymbol{n}_t)_{i,j}$	the number of comparisons of arm i with arm j until time t
$(\hat{\boldsymbol{q}}_t)_{i,j}$	$(\boldsymbol{w}_t)_{i,j} / (\boldsymbol{n}_t)_{i,j}$
α, β	desired errors of type I and II, resp.; in the symmetric case $\alpha = \beta$ we write $\gamma := \alpha = \beta$
G	a digraph
$\mathcal{G}_m$	set of digraphs G on $[m]$, where for all distinct $i, j \in [m]$ either $i \to j$ or $j \to i$ in G
$\overline{\mathcal{G}}_m$	set of tournaments on $[m]$
$\overline{\mathcal{G}}_m(\mathrm{CW})$	set of tournaments on $[m]$, which have a Condorcet winner
$\overline{\mathcal{G}}_m(i^*)$	set of tournaments on $[m]$, which have i^* as Condorcet winner
$\overline{\mathcal{G}}_m(\neg X)$	$\overline{\mathcal{G}}_m \setminus \overline{\mathcal{G}}_m(X)$; here, $X \in \{\mathrm{CW}, i^*\}$
$\mathrm{CW}(G)$	the CW of a tournament $G \in \overline{\mathcal{G}}_m(\mathrm{CW})$; only defined for $G \in \overline{\mathcal{G}}_m(\mathrm{CW})$
$\mathcal{G}_m(X)$	the set $\{G \in \mathcal{G}_m \mid$ every extension G' of G fulfills $G' \in \overline{\mathcal{G}}_m(X)\}$; here, $X \in \{\mathrm{CW}, \neg\mathrm{CW}, i^*, \neg i^*\}$
$\mathcal{G}_m(\Delta i^*), \mathcal{G}_m(\diamondsuit)$	the sets $\mathcal{G}_m(\neg\mathrm{CW}) \cup \bigcup_{j \in [m]: j \neq i^*} \mathcal{G}_m(j)$ and $\bigcup_{i \in [m]} \mathcal{G}_m(i)$, respectively.
$\mathcal{A}_{\mathrm{Bin}}$	a deterministic sequential testing algorithm (DSTA)
$\mathfrak{A}_m$	set of DSTAs for the testification problem; $\mathfrak{A}_m^{\textbf{Verify\_i\_as\_}\mathrm{CW}}$ and $\mathfrak{A}_m^{\textbf{Check\_}\mathrm{CW}}$ are defined accordingly
$T_G^{\mathcal{A}_{\mathrm{Bin}}}$	termination time of the DSTA $\mathcal{A}_{\mathrm{Bin}} \in \mathfrak{A}_m$ when started on G

$T^{\mathcal{A}_{\mathrm{Bin}}}$	worst-case termination time of $\mathcal{A}_{\mathrm{Bin}}$; for testification (without the $\exists$CW-assumption) we have $T^{\mathcal{A}_{\mathrm{Bin}}} = \max_{G\in\overline{\mathcal{G}}_m} T_G^{\mathcal{A}_{\mathrm{Bin}}}$, for other cases cf. Section E
$G_{i\leftrightarrow j}$	digraph defined via $E_{G_{i\leftrightarrow j}} = (E_G \setminus \{(i,j)\}) \cup \{(j,i)\}$
$i_G^{\mathcal{A}_{\mathrm{Bin}}}(t), j_G^{\mathcal{A}_{\mathrm{Bin}}}(t)$	distinct items compared by $\mathcal{A}_{\mathrm{Bin}}$ at time t when started on G
$\{i,j\}_G$	$\{i,j\}_G = (i,j)$ if $i \xrightarrow{G} j$, otherwise $= (j,i)$ (Here: $G \in \overline{\mathcal{G}}_m$, $i,j \in [m]$ distinct)
$\mathfrak{G}_G^{\mathcal{A}_{\mathrm{Bin}}}(t)$	"picture", which $\mathcal{A}$ has of G at time t. Formally, $\mathfrak{G}_G^{\mathcal{A}}(t) \in \mathcal{G}_m$ is defined via $E_{\mathfrak{G}_G^{\mathcal{A}}(t)} = \bigcup_{t'\le t-1}\{\{i_G^{\mathcal{A}}(t'), j_G^{\mathcal{A}}(t')\}_G\}$
$\mathcal{A}_{\mathrm{Bin}}(G)$	output of the DSTA $\mathcal{A}_{\mathrm{Bin}}$ when started on G
Π	set of all sampling strategies
Π_∞	set of $\pi \in \Pi$, which ensure $\lim_{t\to\infty}(\boldsymbol{n}_t)_{i,j} = \infty$ a.s. $\forall (i,j) \in (m)_2$
$C_{h,\gamma'}(n)$	the value $\frac{1}{2n}\left\lceil \ln((1-\gamma')/\gamma') / \ln((1/2+h)/(1/2-h)) \right\rceil$
$\Delta_{(m)_2}$	the set of all $\mathbf{v} = (v_{i,j})_{1\le i<j\le m}$ with $\min_{i<j} v_{i,j} \ge 0$ and $\sum_{i<j} v_{i,j} = 1$
$d_{\mathrm{KL}}(p,q)$	the KL-divergence between two independent random variables $X \sim \mathrm{Ber}(p)$ and $Y \sim \mathrm{Ber}(q)$, i.e., $d_{\mathrm{KL}}(p,q) = p\ln(p/q) + (1-p)\ln((1-p)/(1-q))$
$D_m^h(\mathbf{Q})$	if $\mathbf{Q} \in \mathcal{Q}_m(\mathrm{X})$, $\mathrm{X} \in \{\neg\mathrm{CW}, 1, \ldots, m\}$: $D_m^h(\mathbf{Q}) = \sup_{\mathbf{v}\in\Delta_{(m)_2}} \inf_{\mathbf{Q}'\in\mathcal{Q}_m^h(\neg\mathrm{X})} \sum_{(i,j)\in(m)_2} v_{i,j} d_{\mathrm{KL}}(q_{i,j}, q'_{i,j})$
$\tilde{D}_m^h(\mathbf{Q})$	if $\mathbf{Q} \in \mathcal{Q}_m(\mathrm{X})$, $\mathrm{X} \in \{\neg\mathrm{CW}, \mathrm{CW}\}$: $\tilde{D}_m^h(\mathbf{Q}) = \sup_{\mathbf{v}\in\Delta_{(m)_2}} \inf_{\mathbf{Q}'\in\mathcal{Q}_m^h(\neg\mathrm{X})} \sum_{(i,j)\in(m)_2} v_{i,j} d_{\mathrm{KL}}(q_{i,j}, q'_{i,j})$

A GRAPH-THEORETICAL PREREQUISITES

We suppose here and throughout the whole paper w.l.o.g. that $m \ge 3$ is fulfilled. For $G \in \mathcal{G}_m$ and disjoint $V_1, V_2 \subseteq [m]$ we use in illustrations a double arrow $V_1 \twoheadrightarrow V_2$ to indicate that G contains all the edges $i_1 \to i_2$ with $i_1 \in V_1, i_2 \in V_2$. For example, the graph $G = ([m], E_G) \in \mathcal{G}_m$ with the set of edges $E_G = \{(1,2),(1,3),(1,4),(2,3),(2,4)\}$ may be illustrated as in the figure to the right.



In addition to $\mathcal{G}_m(i^*)$, $\mathcal{G}_m(\mathrm{CW})$ and $\mathcal{G}_m(\neg\mathrm{CW})$ let us also define for any $i^* \in [m]$ the set

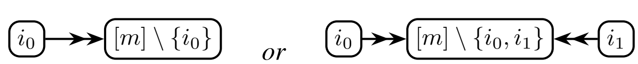
$$\overline{\mathcal{G}}_m(\neg i^*) := \overline{\mathcal{G}}_m \setminus \overline{\mathcal{G}}_m(i^*) = \overline{\mathcal{G}}_m(\neg\mathrm{CW}) \cup \bigcup_{j\in[m]\setminus\{i^*\}} \overline{\mathcal{G}}_m(j)$$

and with this

$$\mathcal{G}_m(\neg i^*) := \{G \in \mathcal{G}_m \mid \text{each extension } G' \text{ of } G \text{ fulfills } G' \in \overline{\mathcal{G}}_m(\neg i^*)\},$$

The following proposition completely characterizes the set $\mathcal{G}_m(\mathrm{CW})$, whereas $\mathcal{G}_m(\neg\mathrm{CW})$, $\mathcal{G}_m(i^*)$ and $\mathcal{G}_m(\neg i^*)$ are completely characterized by Proposition A.3 and Lemma A.4.

Proposition A.1. *For $G \in \mathcal{G}_m$ we have $G \in \mathcal{G}_m(\mathrm{CW})$ iff there exists some $i_0, i_1 \in [m]$ such that G contains at least one of the subgraphs*



In other words, $G \in \mathcal{G}_m(\mathrm{CW})$ is fulfilled iff at least one of the following holds:

(a) There exists $i_0 \in [m]$ such that $i_0 \xrightarrow{G} j$ holds for every $j \in [m] \setminus \{i_0\}$.

(b) There exist distinct $i_0, i_1 \in [m]$ with $i_0 \xrightarrow{G} j$ and $i_1 \xrightarrow{G} j$ for every $j \in [m] \setminus \{i_0, i_1\}$.

In particular, $|E_G| \ge m-1$ holds for every $G \in \mathcal{G}_m(\mathrm{CW})$.

In order to prove Proposition A.1, we at first need some prerequisites. Given some $G \in \overline{\mathcal{G}}_m$ and distinct $i,j \in [m]$ we define $G_{i\leftrightarrow j} \in \overline{\mathcal{G}}_m$ to be the tournament in which the edge between i and j is reversed (in comparison to G) and all the other edges are the same, i.e., if $(i,j) \in E_G$ then

$$E_{G_{i\leftrightarrow j}} = (E_G \setminus \{(i,j)\}) \cup \{(j,i)\}.$$

Note in particular that $G_{i\leftrightarrow j}$ is *not* the graph G with nodes i and j interchanged.

Lemma A.2. *If $G \in \overline{\mathcal{G}}_m(\mathrm{CW})$, $i_0 := \mathrm{CW}(G)$ and $i_1 \in [m] \setminus \{i_0\}$ are such that $G' := G_{i_0 \leftrightarrow i_1} \in \overline{\mathcal{G}}_m(\mathrm{CW})$, then $i_1 = \mathrm{CW}(G')$ holds.*

Proof of Lemma A.2. For every $j \in [m] \setminus \{i_0, i_1\}$ we can infer from $i_0 = \mathrm{CW}(G)$ that $i_0 \xrightarrow{G} j$ and thus also $i_0 \xrightarrow{G'} j$ hold. Together with $i_1 \xrightarrow{G'} i_0$ this shows $\mathrm{CW}(G') \notin [m] \setminus \{i_1\}$, and thus further $\mathrm{CW}(G') = i_1$. □

With this, we are able to prove Proposition A.1.

Proof of Proposition A.1. To show "$\Rightarrow$" indirectly, assume that there was some $G \in \mathcal{G}_m(\mathrm{CW})$ such that neither (a) nor (b) holds. Choose an arbitrary extension $G_0 \in \overline{\mathcal{G}}_m$ of G and note that $G \in \mathcal{G}_m(\mathrm{CW})$ implies that $i_0 := \mathrm{CW}(G_0)$ is well-defined. As (a) does not hold, there exists some $i_1 \in [m] \setminus \{i_0\}$ with $\neg(i_0 \xrightarrow{G} i_1)$. Moreover, the definition of i_0 ensures[1] $\neg(i_1 \xrightarrow{G} i_0)$. Consequently, $G_1 := (G_0)_{i_0 \leftrightarrow i_1} \in \overline{\mathcal{G}}_m$ is also an extension of G, and by assumption on G we have $G_1 \in \overline{\mathcal{G}}_m(\mathrm{CW})$. Thus, we can infer $i_1 = \mathrm{CW}(G_1)$ from Lemma A.2. Since (b) does not hold, there exist $b \in \{0, 1\}$ and $k \in [m] \setminus \{i_0, i_1\}$ such that $\neg(i_b \xrightarrow{G} k)$ holds. From $i_b \xrightarrow{G_b} k$ we can infer $\neg(k \xrightarrow{G} i_b)$. As we have seen that G does neither contain any edge between i_b and k nor between i_{1-b} and i_b and G_{1-b} is an extension of G, also the graph

$$G' := \begin{cases} (G_{1-b})_{i_{1-b} \leftrightarrow i_b} = G_b, & \text{if } k \xrightarrow{G_{1-b}} i_b, \\ \left((G_{1-b})_{i_{1-b} \leftrightarrow i_b}\right)_{i_b \leftrightarrow k}, & \text{if } i_b \xrightarrow{G_{1-b}} k, \end{cases}$$

is an extension of G. Due to $G \in \mathcal{G}_m(\mathrm{CW})$ we obtain $G' \in \overline{\mathcal{G}}_m(\mathrm{CW})$, whence Lemma A.2 guarantees $\mathrm{CW}(G') \in \{i_b, i_{1-b}, k\}$. This is a contradiction, since G' contains by its definition the edges $i_{1-b} \to k$, $k \to i_b$ and $i_b \to i_{1-b}$.

It remains to show "$\Leftarrow$". For this, suppose $G \in \mathcal{G}_m$ to be such that (a) or (b) holds and let $G' \in \overline{\mathcal{G}}_m$ be an arbitrary extension of G. In case (a) holds, we obtain for each $j \in [m] \setminus \{i_0\}$ due to $i_0 \xrightarrow{G} j$ that $i_0 \xrightarrow{G'} j$ is fulfilled. Thus, we can infer $\mathrm{CW}(G') = i_0$ and in particular $G' \in \overline{\mathcal{G}}_m(\mathrm{CW})$. In case (b) holds, G' contains all the edges $i_0 \to j$, $i_1 \to j$, $j \in [m] \setminus \{i_0, i_1\}$. Moreover, for one $b \in \{0, 1\}$ we have $i_b \xrightarrow{G'} i_{1-b}$ and we obtain $i_b = \mathrm{CW}(G')$, i.e., $G' \in \overline{\mathcal{G}}_m(\mathrm{CW})$. □

Proposition A.3. *For $G \in \mathcal{G}_m$ we have the equivalence*

$$G \in \mathcal{G}_m(\neg\mathrm{CW}) \quad \Leftrightarrow \quad \forall\, i \in [m]\, \exists\, j \in [m] \setminus \{i\} : j \xrightarrow{G} i.$$

In particular, $|E_G| \geq m$ holds for every $G \in \mathcal{G}_m(\neg\mathrm{CW})$.

Proof of Proposition A.3. Let $G \in \mathcal{G}_m$ be fixed. To see "$\Leftarrow$" suppose that there is for all $i \in [m]$ some $j = j(i) \in [m] \setminus \{i\}$ with $j \xrightarrow{G} i$ and let G' be an arbitrary extension of G. Then, for any $i \in [m]$, $j(i) \xrightarrow{G'} i$ shows that i can not be the Condorcet winner of G'. We infer $G' \in \overline{\mathcal{G}}_m(\neg\mathrm{CW})$, and arbitrariness of G' lets us conclude $G \in \mathcal{G}_m(\neg\mathrm{CW})$.

To show "$\Rightarrow$" we prove its contraposition. Thus, let us suppose there exists some $i \in [m]$ such that $\neg(j \xrightarrow{G} i)$ holds for every $j \in [m] \setminus \{i\}$. Then, we can choose an extension G' of G with $i \xrightarrow{G'} j$ for every $j \in [m] \setminus \{i\}$. Thus, $G' \in \overline{\mathcal{G}}_m(\mathrm{CW})$ holds with $i = \mathrm{CW}(G')$, which implies $G \notin \mathcal{G}_m(\neg\mathrm{CW})$. □

Lemma A.4. *If $G \in \mathcal{G}_m$ and $i^* \in [m]$, then*

$$G \in \mathcal{G}_m(i^*) \Leftrightarrow \forall j \in [m] \setminus \{i^*\} : i^* \xrightarrow{G} j \quad \textit{and} \quad G \in \mathcal{G}_m(\neg i^*) \Leftrightarrow \exists j \in [m] : j \xrightarrow{G} i^*.$$

In particular, $|E_G| \geq m - 1$ holds for every $G \in \mathcal{G}_m(i^*)$, and $|E_G| \geq 1$ holds for every $G \in \mathcal{G}_m(\neg i^*)$.

Proof. Let $G \in \mathcal{G}_m$ be fixed. For showing "$\Rightarrow$" indirectly suppose there was some $j \in [m] \setminus \{i^*\}$ with $\neg(i^* \xrightarrow{G} j)$. Then, there exists an extension $G' \in \overline{\mathcal{G}}_m$ of G with $j \xrightarrow{G'} i^*$, which is trivially not in $\overline{\mathcal{G}}_m(i^*)$. Thus, we would obtain that $G \notin \mathcal{G}_m(i^*)$, which is a contradiction.

[1] In fact, assuming $i_1 \to i_0$ in G would also imply $i_1 \to i_0$ in G_0, which is according to $i_0 = \mathrm{CW}(G_0)$ not possible.

In order to see "⇐" suppose on the contrary $G \notin \mathcal{G}_m(i^*)$. Then, there exists some extension $G' \in \overline{\mathcal{G}}_m$ of G with $G' \notin \overline{\mathcal{G}}_m(i^*)$. Now, $i^* \neq \mathrm{CW}(G')$ implies the existence of some $j \in [m]$ with $j \xrightarrow{G'} i^*$, and as G' is an extension of G this shows $\neg(i^* \xrightarrow{G} j)$. □

The following Lemma will be crucial for the proofs of Theorems 5.2 and F.3. It allows us to project graphs in $\mathcal{G}_m(\mathrm{CW})$, $\mathcal{G}_m(\neg\mathrm{CW})$, $\mathcal{G}_m(i^*)$, $\mathcal{G}_m(\neg i^*)$ as well as in

$$\mathcal{G}_m(\Delta i^*) := \mathcal{G}_m(\neg\mathrm{CW}) \cup \bigcup_{j\in[m]:j\neq i^*} \mathcal{G}_m(j) \quad \text{and} \quad \mathcal{G}_m(\diamondsuit) := \bigcup_{i\in[m]} \mathcal{G}_m(i)$$

to characteristic subgraphs, respectively. Note here that $\mathcal{G}_m(\Delta i^*) \neq \mathcal{G}_m(\neg i^*)$, as for instance the graph $([m], \{(2,1)\})$ is contained in $\mathcal{G}_m(\neg 1)$ but not in $\mathcal{G}_m(\Delta 1)$. According to Proposition A.1, $\mathcal{G}_m(\mathrm{CW}) \supsetneq \mathcal{G}_m(\diamondsuit)$, so that these notions are not redundant.

Lemma A.5. *Let $i^* \in [m]$. There exist mappings $\mathfrak{l}_{\mathrm{CW}}, \mathfrak{l}_{\neg\mathrm{CW}}, \mathfrak{l}_{i^*}, \mathfrak{l}_{\neg i^*}, \mathfrak{l}_{\Delta i^*}, \mathfrak{l}_{\diamondsuit} : \mathcal{G}_m \to \mathcal{G}_m$ with the following properties:*

(a) $E_{\mathfrak{l}_{\mathrm{CW}}(G)}, E_{\mathfrak{l}_{\neg\mathrm{CW}}(G)}, E_{\mathfrak{l}_{i^*}(G)}, E_{\mathfrak{l}_{\neg i^*}(G)}, E_{\mathfrak{l}_{\Delta i^*}(G)}, E_{\mathfrak{l}_{\diamondsuit}(G)} \subseteq E_G$ *for every* $G \in \mathcal{G}_m$*,*

(b) $|E_{\mathfrak{l}_{\mathrm{CW}}(G)}| \in \{0, m-1, 2m-4\}$, $|E_{\mathfrak{l}_{\neg\mathrm{CW}}(G)}| \in \{0, m\}$, $|E_{\mathfrak{l}_{i^*}(G)}| \in \{0, m-1\}$, $|E_{\mathfrak{l}_{\neg i^*}(G)}| \in \{0, 1\}$, $|E_{\mathfrak{l}_{\Delta i^*}(G)}| \in \{0, m-1, m\}$ *and* $|E_{\mathfrak{l}_{\diamondsuit}(G)}| \in \{0, m-1\}$ *for every* $G \in \mathcal{G}_m$*,*

(c) for every $G \in \mathcal{G}_m$ *we have the equivalences*

$$G \in \mathcal{G}_m(\mathrm{CW}) \Leftrightarrow \mathfrak{l}_{\mathrm{CW}}(G) \in \mathcal{G}_m(\mathrm{CW}), \quad G \in \mathcal{G}_m(\neg\mathrm{CW}) \Leftrightarrow \mathfrak{l}_{\neg\mathrm{CW}}(G) \in \mathcal{G}_m(\neg\mathrm{CW}),$$
$$G \in \mathcal{G}_m(i^*) \Leftrightarrow \mathfrak{l}_{i^*}(G) \in \mathcal{G}_m(i^*), \quad G \in \mathcal{G}_m(\neg i^*) \Leftrightarrow \mathfrak{l}_{\neg i^*}(G) \in \mathcal{G}_m(\neg i^*),$$
$$G \in \mathcal{G}_m(\Delta i^*) \Leftrightarrow \mathfrak{l}_{\Delta i^*}(G) \in \mathcal{G}_m(\Delta i^*), \quad G \in \mathcal{G}_m(\diamondsuit) \Leftrightarrow \mathfrak{l}_{\diamondsuit}(G) \in \mathcal{G}_m(\diamondsuit).$$

Proof. To define $\mathfrak{l}_{\mathrm{CW}}$ suppose $G \in \mathcal{G}_m$ to be fixed for the moment. In case $G \notin \mathcal{G}_m(\mathrm{CW})$ we may simply define $\mathfrak{l}_{\mathrm{CW}}(G) := ([m], \emptyset)$, and in case $G \in \mathcal{G}_m(\mathrm{CW})$ there exist according to Proposition A.1 two distinct $i_0, i_1 \in [m]$ such that at least one of

$$E[i_0] := \{(i_0, j) \,:\, j \in [m] \setminus \{i_0\}\}$$

and

$$E[i_0; i_1] := \{(i_0, j) \,:\, j \in [m] \setminus \{i_0, i_1\}\} \cup \{(i_1, j) \,:\, j \in [m] \setminus \{i_0, i_1\}\}$$

is a subset of E_G, i.e., we may define

$$\mathfrak{l}_{\mathrm{CW}}(G) := \begin{cases} ([m], E[i_0]), & \text{if } E[i_0] \subset E_G, \\ ([m], E[i_0; i_1]), & \text{otherwise.} \end{cases}$$

It is straightforward to check that $\mathfrak{l}_{\mathrm{CW}}$ fulfills all the desired properties.

The existence of $\mathfrak{l}_{\neg\mathrm{CW}}$, $\mathfrak{l}_{i^*}$ and $\mathfrak{l}_{\neg i^*}$ follow from Proposition A.3 and Lemma A.4.

For defining $\mathfrak{l}_{\Delta i^*}$ let $G \in \mathcal{G}_m$ be given. In case $G \notin \mathcal{G}_m(\Delta i^*)$ we define $\mathfrak{l}_{\Delta i^*}(G) := ([m], \emptyset)$. In the remaining case $G \in \mathcal{G}_m(\Delta i^*)$ we choose

$$\mathfrak{l}_{\Delta i^*}(G) := \begin{cases} \mathfrak{l}_j(G), & \text{if } \exists j \in [m] \setminus \{i^*\} \text{ with } G \in \mathcal{G}_m(j), \\ \mathfrak{l}_{\neg\mathrm{CW}}(G), & \text{otherwise.} \end{cases}$$

Note that this is due to $\mathcal{G}_m(j) \cap \mathcal{G}_m(j') = \emptyset$ for $j \neq j'$ well-defined. Then, $\mathfrak{l}_{\Delta i^*}$ has all the properties stated above.

Finally, we define $\mathfrak{l}_{\diamondsuit}$ via

$$\mathfrak{l}_{\diamondsuit}(G) := \begin{cases} \mathfrak{l}_j(G), & \text{if } \exists j \in [m] \text{ with } G \in \mathcal{G}_m(j), \\ ([m], \emptyset), & \text{otherwise.} \end{cases}$$

□

B PROBABILISTIC PREREQUISITES

In this section, we discuss the sample complexity of testing whether the bias p of an unfair coin is greater or smaller than $1/2$. To formalize this, suppose $p \in [0,1]$ to be fixed but unknown to us and let $\{X_n\}_{n\in\mathbb{N}}$ be a sequence of iid samples $X_n \sim \mathrm{Ber}(p)$, $n \in \mathbb{N}$, which are w.l.o.g. defined on the same probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Here, we write $\mathbb{P}_p(A)$ for the probability of an event A which is an element of the σ-algebra generated by $(X_n)_{n\in\mathbb{N}}$ for X_n iid with distribution $\mathrm{Ber}(p)$. Moreover, we write $\mathbb{E}_p$ for the expectation w.r.t. $\mathbb{P}_p$. We are interested in deciding

$$\mathbf{H}_0 : p > 1/2 \quad \text{versus} \quad \mathbf{H}_1 : p < 1/2. \tag{1}$$

The following result justifies the choice of $C_{h,\gamma}(n)$ as $\frac{1}{2n}\left\lceil \frac{\ln((1-\gamma)/\gamma)}{\ln(\left(1/2+h\right)/\left(1/2-h\right))} \right\rceil$ in Algorithms 1, 5 and 6.

Lemma B.1. *Suppose $0 < \gamma < \gamma_0 < 1/2$ and $0 < h < h_0 < 1/2$ to be fixed.*

(i) Let $\mathcal{A}$ be the algorithm, which samples $X_1, X_2, \dots$ until the first time n, where $\frac{1}{n}\sum_{k=1}^n X_k \notin [1/2 \pm C_{h,\gamma}(n)]$ and decides for 0 in case $\frac{1}{n}\sum_{k=1}^n X_k > 1/2 + C_{h,\gamma}(n)$ and for 1 in case $\frac{1}{n}\sum_{k=1}^n X_k < 1/2 - C_{h,\gamma}(n)$. Then, $\mathcal{A}$ decides (1) with error probability at most γ for any $p \in [0, 1/2-h] \cup [1/2+h, 1]$, i.e., we have

$$\forall p \geq 1/2 + h : \mathbb{P}_p(\mathbf{D}(\mathcal{A}) = 0) \geq 1 - \gamma \quad \textit{and} \quad \forall p \leq 1/2 - h : \mathbb{P}_p(\mathbf{D}(\mathcal{A}) = 1) \geq 1 - \gamma.$$

Moreover, the termination time $T^{\mathcal{A}}$ of $\mathcal{A}$ fulfills

$$\sup_{p\in[0,1/2-h]\cup[1/2+h,1]} \mathbb{E}_p[T^{\mathcal{A}}] = \mathbb{E}_{1/2\pm h}[T^{\mathcal{A}}] = (2h)^{-1}\left\lceil \frac{\ln((1-\gamma)/\gamma)}{\ln(\left(1/2+h\right)/\left(1/2-h\right))} \right\rceil (1-2\gamma), \tag{2}$$

which is in $\mathcal{O}(h^{-2}\ln(\gamma^{-1}))$ as $\max\{h^{-1},\gamma^{-1}\} \to \infty$.

(ii) The algorithm $\mathcal{A}$ from (i) is w.r.t. $\mathbb{E}_{1/2+h}[T^{\mathcal{A}}]$ and $\mathbb{E}_{1/2-h}[T^{\mathcal{A}}]$ optimal among all algorithms, which decide (1) with error probability at most γ for any $p \in \{1/2 \pm h\}$. In other words: If $\mathcal{A}'$ is an algorithm, which fulfills

$$\mathbb{P}_{1/2+h}(\mathbf{D}(\mathcal{A}') = 0) \geq 1 - \gamma \quad \textit{and} \quad \mathbb{P}_{1/2-h}(\mathbf{D}(\mathcal{A}') = 1) \geq 1 - \gamma,$$

then it fulfills

$$\mathbb{E}_{1/2\pm h}[T^{\mathcal{A}'}] \geq \mathbb{E}_{1/2\pm h}[T^{\mathcal{A}}] \geq c(h_0,\gamma_0) h^{-2}\ln(\gamma^{-1})$$

for some appropriate positive constant $c(h_0,\gamma_0)$, which does not depend on γ or h.

Proof. (i) The test $\mathcal{A}$ from (i) is the sequential probability ratio test (SPRT) for the problem at hand and has its origins in [Wald, 1945]. Statement (i) can be inferred from p.10–15 in [Siegmund, 1985]. To see this, define $c := \left\lceil \frac{\ln((1-\gamma)/\gamma)}{\ln(\left(1/2+h\right)/\left(1/2-h\right))} \right\rceil$, write $S_n := \sum_{k=1}^n (2X_k - 1)$ and note that $\mathcal{A}$ is exactly the test which samples until $S_n \leq -c$ or $S_n \geq c$ and decides for 0 in the first and for 1 in the second case. Now, equation (2.28) on p.15 in [Siegmund, 1985] shows that

$$\mathbb{P}_{1-p}\left(S_{T^{\mathcal{A}}} \leq -c\right) = \mathbb{P}_p\left(S_{T^{\mathcal{A}}} \geq c\right) = \left(1 + {(1-p)^c}/{p^c}\right)^{-1}$$

for every $p \neq 1/2$. Since c is chosen such that the right-hand side is $\leq \gamma$ if $p = 1/2 - h$, $\mathcal{A}$ is able to decide (1) with error $\leq \gamma$ for $p \in \{1/2 \pm h\}$. As $p \mapsto 1/(1 + (1-p)^c/p^c)$ is monotonically increasing on $[1/2, 1]$, $\mathcal{A}$ decides (1) with error probability at most γ also for every $p \in [0, 1/2-h) \cup (1/2+h, 1]$. Moreover, equation (2.29) on p.15 in [Siegmund, 1985] shows that for each $h' \in (0, 1/2)$

$$\mathbb{E}_{1/2\pm h'}[T^{\mathcal{A}}] = (2h')^{-1} c \left| 1 - 2\left(1 + \left({(1/2+h')^c}/{(1/2-h')^c}\right)\right)^{-1} \right|, \tag{3}$$

which is continuous and decreasing in h' for $h' \in (0, 1/2)$. Consequently, (2) holds by the choice of c. Using that $\frac{x}{x+1} < \ln(1+x) < x$ holds for each $x > -1$ we see that $\ln(\left(1/2+h\right)/\left(1/2-h\right)) \in \Theta(h)$ as $h \to 0$, and thus the right-hand side of (2) is in $\mathcal{O}(h^{-2}\ln(\gamma^{-1}))$ as $\min\{h,\gamma\} \to 0$.

(ii) For the optimality of $\mathcal{A}$ stated in (ii) as a solution for deciding (1) with error $\leq \gamma$ for $p \in \{1/2 \pm h\}$ confer pages 19–22 in [Siegmund, 1985] or [Ferguson, 1967, Theorem 2, pp. 365] or the original proof in [Wald and Wolfowitz, 1948].

In order to conclude the lemma, we need to show a lower bound for the right-hand side of (2) of the form $c(h_0, \gamma_0)h^{-2}\ln(\gamma^{-1})$ for some appropriate constant $c(h_0, \gamma_0)$, which does not depend on γ or h. The function $f : (0,1) \to \mathbb{R}, \gamma \mapsto \frac{\ln((1-\gamma)/\gamma)\cdot(1-2\gamma)}{\ln(1/\gamma)}$ fulfills $f(1/2) = 0$ and

$$f'(\gamma) = \frac{(1-2\gamma)\ln(\gamma^{-1}) - (\gamma-1)\ln(\gamma^{-1}-1)(2\gamma + 2\gamma\ln(\gamma^{-1}) - 1))}{(\gamma-1)\gamma\ln^2(\gamma^{-1})} < 0$$

for every $\gamma \in (0, 1/2)$. Consequently, there exists some $c'(\gamma_0) > 0$ with $\ln((1-\gamma)/\gamma)(1-2\gamma) \geq c'(\gamma_0)\ln(1/\gamma)$ for each $\gamma \in (0, \gamma_0)$. Moreover, as $\ln(1+x) < x$ for $x > -1$, we obtain for $h \in (0, h_0)$ the inequality

$$\ln\left(\frac{1/2+h}{1/2-h}\right) = \ln\left(1 + \frac{4h}{1-2h}\right) < \frac{4h}{1-2h} < \frac{4h}{1-2h_0}.$$

Combining these estimates, we obtain with $c(h_0, \gamma_0) := \frac{c'(\gamma_0)(1-2h_0)}{8}$ that

$$(2h)^{-1}\left\lceil \frac{\ln((1-\gamma)/\gamma)}{\ln\left(\frac{1/2+h}{1/2-h}\right)} \right\rceil (1-2\gamma) \geq c(h_0, \gamma_0)h^{-2}\ln(\gamma^{-1}).$$

As the SPRT is optimal (w.r.t. expected runtime) this shows that any such algorithm $\mathcal{A}'$ as in (ii) fulfills

$$\mathbb{E}_{1/2\pm h}[T^{\mathcal{A}'}] \geq c(h_0, \gamma_0)h^{-2}\ln(\gamma^{-1}).$$

□

C SELECT-THEN-VERIFY

In this section, we provide the details for $\mathcal{A}$-THEN-VERIFY presented in the beginning of Section 5, with a special emphasis on the case where $\mathcal{A}$ is the algorithm SELECT in [Mohajer et al., 2017]. At first sight, our framework may resemble the EXPLORE-VERIFY FRAMEWORK in [Karnin, 2016], but in contrast to ours the latter can only be used to find the CW provided it exists and does **not** to solve the CW testification task. For the sake of convenience, we focus again on the symmetric case $\alpha = \beta =: \gamma$, write $\tilde{\mathcal{O}}$ for $\mathcal{O}$, which hides $\ln(m)$-factors, and use notation, which is introduced in the beginning of Section F.

Algorithm 1 $\mathcal{A}$-THEN-VERIFY

Input: m, h, γ

1: $i \leftarrow \mathcal{A}(m, h, \gamma/2)$
2: **for** $j \in [m] \setminus \{i\}$ **do**
3: Conduct a test T for $\mathbf{H}_0 : q_{i,j} > 1/2 + h$ versus $\mathbf{H}_1 : q_{i,j} < 1/2 - h$ with an error $\leq \frac{\gamma}{2(m-1)}$.
4: **if** T decides for $\mathbf{H}_1$ **then** **return** $\neg$CW
5: **return** i

Proposition C.1. *If $\mathcal{A}$ solves **Testification**$\{\exists\text{CW}\}$ on $\mathcal{Q}_m^h$ for $\frac{\gamma}{2}, \frac{\gamma}{2}$, then Algorithm 1 solves $\mathcal{P}^{m,h,\gamma,\gamma}$.*

Proof sketch. Suppose $\mathcal{A}$ to be a fixed solution of **Testification**$\{\exists\text{CW}\}$ on $\mathcal{Q}_m^h$ for $\frac{\gamma}{2}, \frac{\gamma}{2}$ and write for convenience $\mathcal{A}'$ for the corresponding version of Algorithm 1 with parameters m, h, γ. Due to a union bound, for any $\mathbf{Q} \in \mathcal{Q}_m^h$, the probability that the sign of any $q_{i,j} - 1/2$, $j \neq i$, is estimated incorrectly in Steps 2–4 of Algorithm 1 is at most γ.

Suppose at first $\mathbf{Q} \in \mathcal{Q}_m^h(\text{CW})$. If $\mathcal{A}'$ makes an error, then the output of $\mathcal{A}$ in step 1 is incorrect or a mistake is made in steps 2–7. Since both of these happen happen with error prob. $\leq \gamma/2$, the overall error of $\mathcal{A}'$ is at most γ. Now, suppose $\mathbf{Q} \in \mathcal{Q}_m^h(\neg\text{CW})$. If an error occurs, the candidate i of $\mathcal{A}$ from Step 1 is falsely verified to fulfill $\min_{j\neq i} q_{i,j} > 1/2$ in lines 2–4, whence

$$\mathbb{P}_{\mathbf{Q}}(\mathbf{D}(\mathcal{A}') \in [m]) = \sum\nolimits_{i\in[m]} \mathbb{P}_{\mathbf{Q}}\left(\text{ an error is made in steps 2–4} \,\middle|\, \mathbf{D}(\mathcal{A}) = i\right) \mathbb{P}_{\mathbf{Q}}(\mathbf{D}(\mathcal{A}) = i) \leq \gamma/2.$$

□

There are numerous possible choices for the test T in Step 3 of Algorithm 1. In the following, we choose T as the corresponding *Sequential Probability Ratio Test* for the hypothesis test, which is according to Lemma B.1 the optimal choice w.r.t. the worst-case expected sample complexity.

The problem of identifying the Condorcet winner is also referred to as the *best-arm-identification* problem in the dueling bandits literature [Karnin, 2016, Bengs et al., 2021]. As many solutions to this problem have stronger requirements than the mere existence of the Condorcet winner, they can not be used without further adaptations as a candidate for $\mathcal{A}$ in Proposition C.1. For example, SEEBS from [Ren et al., 2020] formally requires *strong stochastic transitivity* (SST) as well as the *stochastic triangle inequality* (STI)[2] to hold; thus, SEEBS is proven to correctly identify the CW with error $\leq \gamma$ only for any $\mathbf{Q}$ in a proper subset $\mathcal{Q}_m^h(\mathrm{SST} \wedge \mathrm{STI}) \subsetneq \mathcal{Q}_m^h(\mathrm{CW})$. As a consequence, the error of SEEBS-THEN-VERIFY could only be guaranteed to be $\leq \gamma$ whenever $\mathbf{Q} \in \mathcal{Q}_m^h(\mathrm{SST} \wedge \mathrm{STI}) \cup \mathcal{Q}_m^h(\neg\mathrm{CW}) \subsetneq \mathcal{Q}_m^h$. In other words, we can **not** infer that SEEBS-THEN-VERIFY solves $\mathcal{P}^{m,h,\gamma,\gamma}$.

Mohajer et al. [2017] present a solution SELECT for the best-arm-identification problem and suppose for its theoretical analysis so-called *weak stochastic transitivity*[2] to hold. Fortunately, it can be shown that this assumption is not necessary and instead the mere existence of a CW suffices, whence SELECT-THEN-VERIFY is a solution to $\mathcal{P}^{m,h,\gamma,\gamma}$. SELECT conducts a knockout-tournament between all m arms, in which two competing arms i, j are dueled for a fixed number of times N and i wins this comparison if it wins at least $N/2$ of the duels. Choosing

$$N := \frac{(1+\varepsilon)\ln(2)\log_2(\log_2(m))}{2h^2} \quad \text{with} \quad \varepsilon := -\frac{\ln(\gamma/2)}{\ln(\log_2(m))},$$

it can be shown that SELECT is a solution to **Testification**$\{\exists\mathrm{CW}\}$ on $\mathcal{Q}_m^h$ for $\frac{\gamma}{2}, \frac{\gamma}{2}$ with constant sample complexity $\lceil (m-1)N \rceil$. With the help of Proposition C.1 and Lemma B.1 we can infer that SELECT-THEN-VERIFY solves $\mathcal{P}^{m,h,\gamma,\gamma}$ with a worst-case expected sample complexity on $\mathcal{Q}_m^h$ of order $\tilde{O}(mh^{-2}\ln(\gamma^{-1}))$ as $\max\{m, h^{-1}, \gamma^{-1}\} \to \infty$, which is with regard to Theorem 4.1 (up to logarithmic terms) asymptotically optimal.

Of course, there may be other candidates for $\mathcal{A}$ in Algorithm 1, e.g., as already indicated, a solution $\mathcal{A}$ to **Testification**$\{\exists\mathrm{CW}\}$ may be inferred from [Karnin, 2016]. For the sake of convenience and simplicity, we have restricted ourselves to SELECT at this point, because the algorithm itself and its theoretical guarantees fit into our setting – e.g., it is implicitly assumed that $\mathbf{Q} \in \mathcal{Q}_m^h$– and thus makes the discussed optimality (up to logarithmic terms) of SELECT-THEN-VERIFY rather easy to see. Even though a theoretical comparison of our solution to SELECT-THEN-VERIFY on arbitrary instances appears infeasible, we are able to show the following result.

Lemma C.2. *Let $m \in \mathbb{N}$, $\gamma \in (0, 1/2)$ be arbitrary. For sufficiently small $h > 0$ and $\tilde{h} \in (h, 1/2)$ we have: Whenever $\mathcal{A}^{\mathrm{NTS}}$ from Corollary 5.4 and $\mathcal{A}'$:=SELECT-THEN-VERIFY are started with parameters m, h, γ, then*

$$\mathbb{E}_{\mathbf{Q}}\left[T^{\mathcal{A}^{\mathrm{NTS}}}\right] \leq \frac{\mathbb{E}_{\mathbf{Q}}[T^{\mathcal{A}'}]}{2}$$

holds for any $\mathbf{Q} \in \mathcal{Q}_m^{\tilde{h}} \subsetneq \mathcal{Q}_m^h$.

Proof. At first, let us recall the corresponding lower and upper bounds for $\mathcal{A}^{\mathrm{NTS}}$ and $\mathcal{A}'$, which we will use. For any $m \in \mathbb{N}, \gamma \in (0,1), h \in (0, 1/2), \tilde{h} \in (h, 1/2)$ and $\mathbf{Q} \in \mathcal{Q}_m \setminus Q_m^{\tilde{h}}$ the instance-wise upper bound from Theorem H.1 (with $\mathcal{A}_{\mathrm{Bin}}$ chosen as in Corollary 5.4) yields

$$\mathbb{E}_{\mathbf{Q}}\left[T^{\mathcal{A}^{\mathrm{NTS}}}\right] \leq \frac{c(h,\gamma')(2m - \lfloor \log_2 m \rfloor - 2)}{2\tilde{h}} \left| 1 - 2\left(1 + \left({}^{(1/2+\tilde{h})}/{}_{(1/2-\tilde{h})}\right)^{c(h,\gamma')}\right)^{-1} \right| =: g(\tilde{h})$$

for any $\mathbf{Q} \in \mathcal{Q}_m^{\tilde{h}}$, where $\gamma' := \gamma/m$ and

$$c(h,\gamma') := \left\lceil \frac{\ln((1-\gamma')/\gamma')}{\ln(1/2+h)/(1/2-h))} \right\rceil.$$

[2] A relation $\mathbf{Q} \in \mathcal{Q}_m$ is called *weakly stochastic transitive* (WST) if $(q_{i,j} \geq 1/2$ and $q_{j,k} \geq 1/2) \Rightarrow q_{i,k} \geq 1/2$ holds for every distinct $i, j, k \in [m]$ and it is said to be *strongly stochastic transitive* if the implication $(q_{i,j} \geq 1/2$ and $q_{j,k} \geq 1/2) \Rightarrow q_{i,k} \geq \max\{q_{i,j}, q_{j,k}\}$ is fulfilled for every distinct $i, j, k \in [m]$. Moreover, a relation $\mathbf{Q} \in \mathcal{Q}_m$, which is WST (or SST), is said to satisfy the *stochastic triangle inequality* if $(q_{i,j} \geq 1/2$ and $q_{j,k} \geq 1/2) \Rightarrow q_{i,k} \leq q_{i,j} + q_{j,k} - 1/2$ holds for any distinct $i, j, k \in [m]$.

Moreover, the "SELECT-part" of SELECT-THEN-VERIFY (started with m, γ, h) requires for any $\mathbf{Q} \in \mathcal{Q}_m$ exactly

$$\frac{(m-1)(1+\epsilon(m,\gamma))\ln(2)\log_2(\log_2 m)}{2h^2} \quad \text{with } \epsilon(m,\gamma) := -\frac{\ln(\gamma/2)}{\ln(\log_2 m)}$$

samples. Note that this is trivially a lower bound for $\mathbb{E}_{\mathbf{Q}}[T^{\mathcal{A}'}]$.

Now, let $m \in \mathbb{N}$ and $\gamma \in (0, 1/2)$ be fixed. Define $\gamma' := \gamma/m$ and choose $h \in (0, 1/2)$ with

$$h < \frac{(m-1)(1+\epsilon(m,\gamma))\ln(2)\log_2(\log_2 m)}{2\ln((1-\gamma')/\gamma')(2m - \lfloor \log_2 m \rfloor - 2)}.$$

Then, $c(h, \gamma')$ is fixed. As $\ln((1/2+\hat{h})/(1/2-\hat{h})) > 4\hat{h}$ holds for any $\hat{h} \in (0, 1/2)$, we have $c(h,\gamma') < \frac{\ln((1-\gamma')/\gamma')}{4h}$. Regarding that $\frac{1}{2\hat{h}} \to 1$ and $\left(\frac{1/2+\hat{h}}{1/2-\hat{h}}\right)^{c(h,\gamma')} \to \infty$ as $\hat{h} \to 1/2$, we obtain

$$g(\hat{h}) \longrightarrow c(h,\gamma')(2m - \lfloor \log_2(m) \rfloor - 2) \leq \frac{\ln((1-\gamma')/\gamma')(2m - \lceil \log_2 m \rceil - 2)}{4h}$$

Consequently, we have for sufficiently large $\tilde{h} \in (h, 1/2)$ and any $\mathbf{Q} \in \mathcal{Q}_m^{\tilde{h}}$

$$\begin{aligned}\mathbb{E}_{\mathbf{Q}}[T^{\mathcal{A}^{\mathrm{NTS}}}] \leq g(\tilde{h}) &\leq \frac{\ln((1-\gamma')/\gamma')(2m - \lceil \log_2 m \rceil - 2)}{2h} \\ &\leq \frac{(m-1)(1+\epsilon(m,\gamma))\ln(2)\log_2(\log_2 m)}{4h^2} \leq \frac{\mathbb{E}_{\mathbf{Q}}[T^{\mathcal{A}'}]}{2},\end{aligned}$$

where the third inequality holds due to the choice of h. □

D PROOF OF THEOREM 5.2

We continue with the proof of Theorem 5.2, which states that Algorithm 1 indeed solves the testification problem for the CW on $\mathcal{Q}_m^h$ for α and β, provided π is a sampling strategy in Π_∞. Note that the correction terms $1/m$ and $1/(m-1)$ associated for the type I/II error probabilities α/β are chosen in an optimal way with regard to a Bonferroni correction due to Proposition A.3 and Lemma A.4, respectively.

Proof of Theorem 5.2. For convenience we abbreviate $T := T^{\mathcal{A}}$ and use the notations $n_{i,j}(t) := (\boldsymbol{n}_t)_{i,j}$, $w_{i,j}(t) := (\boldsymbol{w}_t)_{i,j}$ and $\hat{q}_{i,j}(t) := (\hat{\boldsymbol{q}}_t)_{i,j}$. Let $\mathbf{Q} \in \mathcal{Q}_m^h$ be fixed for the moment. With regard to the definition of $\mathcal{A}$, there exists an independent family $\{x_n^{[q_{i,j}]}\}_{n \in \mathbb{N}_0, 1 \leq i < j \leq m}$ of random variables $x_n^{[q_{i,j}]} \sim \mathrm{Ber}(q_{i,j})$ such that $w_{i,j}(t) \stackrel{d}{=} \sum_{n=1}^{n_{i,j}(t)} x_n^{[q_{i,j}]}$ and $\hat{q}_{i,j}(t) \stackrel{d}{=} \frac{1}{n_{i,j}(t)} \sum_{n=1}^{n_{i,j}(t)} x_n^{[q_{i,j}]}$ hold for every $t \in \mathbb{N}_0$[3]. Recall that $\gamma' = \min\{\frac{\alpha}{m}, \frac{\beta}{m-1}\}$. We split the remaining proof into four parts.

Part 1: Almost sure finiteness of T

By the strong law of large numbers and the assumption $\pi \in \Pi_\infty$, we have $n_{i,j}(t) \to \infty$ and $\hat{q}_{i,j}(t) \to q_{i,j} \in [0, \frac{1}{2} - h) \cup (\frac{1}{2} + h, 1]$ almost surely as $t \to \infty$ for each $(i,j) \in (m)_2$. Together with $C_{h,\gamma'}(n) \to 0$ as $n \to \infty$ we obtain that

$$T' := \min\{t \in \mathbb{N} \,|\, \hat{q}_{i,j}(t) \notin [1/2 \pm C_{h,\gamma'}(n_{i,j}(t))] \text{ for all distinct } i,j \in [m]\}$$

is almost surely finite. Regarding the definitions of $\mathcal{A}$ and T' we see that $\hat{G}_{T'}$ is almost surely an element of $\overline{\mathcal{G}}_m = \bigcup_{i^* \in [m]} \overline{\mathcal{G}}_m(i^*) \cup \overline{\mathcal{G}}_m(\neg\mathrm{CW})$, i.e., $\hat{G}_{T'}$ is with probability 1 either in $\overline{\mathcal{G}}_m(\neg\mathrm{CW}) \subsetneq \mathcal{G}_m(\neg\mathrm{CW})$ or in $\overline{\mathcal{G}}_m(i^*) \subsetneq \mathcal{G}_m(i^*)$ for some $i^* \in [m]$. Consequently, we obtain

$$T = \min\left\{t \in \mathbb{N} : \hat{G}_t \in \mathcal{G}_m(\neg\mathrm{CW}) \text{ or } \hat{G}_t \in \mathcal{G}_m(i^*) \text{ for some } i^* \in [m]\right\} \leq T' < \infty \quad \text{a.s.},$$

which completes the proof of Part 1.

[3] $\stackrel{d}{=}$ denotes equality in distribution

Before showing the guarantees on the type I and II error we fix some further notation: For $\mathbf{Q} \in \mathcal{Q}_m$, $E \subseteq [m] \times [m]$ and $\{i,j\} \in [m]_2$ we say that $\{i,j\}$ **is assigned incorrectly in** E **w.r.t.** $\mathbf{Q}$ if

$$(i,j) \in E \text{ and } q_{i,j} < 1/2 \quad \text{or} \quad (j,i) \in E \text{ and } q_{i,j} > 1/2$$

holds, where we may omit the term "w.r.t. $\mathbf{Q}$" in case $\mathbf{Q}$ is clear from the context.

Part 2: Showing $\mathbb{P}_{\mathbf{Q}}(\{i,j\}$ is assigned incorrectly in $\hat{E}_T) \leq \gamma'$ in case $|q_{i,j} - 1/2| > h$

Let $(i,j) \in (m)_2$ be fixed and define the stopping time

$$T_{(i,j)} := \min\{t \in \mathbb{N} \,:\, t \leq T \text{ and } |\hat{q}_{i,j}(t) - 1/2| > C_{h,\gamma'}(n_{i,j}(t))\} \in \mathbb{N} \cup \{\infty\}.$$

A look at lines 8 and 10 of Algorithm 1 reveals $\hat{E}_{t-1} \subseteq \hat{E}_t$ for all $t \leq T$ and moreover

$$(i,j) \in \hat{E}_T \quad \Leftrightarrow \quad \left(T_{(i,j)} < \infty \text{ and } \hat{q}_{i,j}(T_{(i,j)}) - 1/2 > C_{h,\gamma'}(n_{i,j}(T_{(i,j)}))\right).$$

Consequently, Lemma B.1 assures in the case $q_{i,j} < 1/2 - h$ that

$$\begin{aligned}\mathbb{P}_{\mathbf{Q}}\left(\{i,j\} \text{ is assigned incorrectly in } \hat{E}_T\right) &= \mathbb{P}_{\mathbf{Q}}((i,j) \in \hat{E}_T) \\ &\leq \mathbb{P}_{\mathbf{Q}}\left(\frac{1}{n_{i,j}(T_{(i,j)})} \sum_{n=1}^{n_{i,j}(T_{(i,j)})} x_n^{[q_{i,j}]} - 1/2 > C_{h,\gamma'}(n_{i,j}(T_{(i,j)}))\right) \leq \gamma',\end{aligned}$$

and in case $q_{j,i} < 1/2 - h$ we similarly obtain

$$\mathbb{P}_{\mathbf{Q}}(\{i,j\} \text{ is assigned incorrectly in } \hat{E}_T) = \mathbb{P}_{\mathbf{Q}}((j,i) \in \hat{E}_T) \leq \gamma'$$

due to symmetry. This shows the assertion of Part 2.

In the following let $\iota_{\Delta i^*}$, $i^* \in [m]$ and $\iota_{\diamondsuit}$ be defined as in Lemma A.5.

Part 3: Bounding the type I error

Suppose $i^* \in [m]$ and $\mathbf{Q} \in \mathcal{Q}_m^h(i^*)$. Part (c) of Lemma A.5 yields the identity

$$\begin{aligned}\mathbb{P}_{\mathbf{Q}}(\mathbf{D}(\mathcal{A}) \neq i^*) &= \mathbb{P}_{\mathbf{Q}}(\mathbf{D}(\mathcal{A}) = \neg\text{CW or } \mathbf{D}(\mathcal{A}) = j \text{ for some } j \in [m] \setminus \{i^*\}) \\ &= \mathbb{P}_{\mathbf{Q}}(\hat{G}_T \in \mathcal{G}_m(\Delta i^*)) = \mathbb{P}_{\mathbf{Q}}(\iota_{\Delta i^*}(\hat{G}_T) \in \mathcal{G}_m(\Delta i^*)). \qquad (4)\end{aligned}$$

For the sake of convenience, we write $\overline{G}$ for the set $\{\{i,j\} \,|\, (i,j) \in E_G \text{ or } (j,i) \in E_G\}$ for $G \in \mathcal{G}_m$. By Part 2 we have

$$\mathbb{P}_{\mathbf{Q}}\left(\{i,j\} \text{ is ass. inc. in } E_{\iota_{\Delta i^*}(\hat{G}_T)} \,\middle|\, \overline{\iota_{\Delta i^*}(\hat{G}_T)} = \overline{G}\right) \leq \begin{cases} 0, & \text{if } \{i,j\} \notin \overline{G}, \\ \gamma', & \text{if } \{i,j\} \in \overline{G}, \end{cases}$$

for any $G \in \mathcal{G}_m$ with $\mathbb{P}_{\mathbf{Q}}(\overline{\iota_{\Delta i^*}(\hat{G}_T)} = \overline{G}) > 0$. If no $\{i,j\} \in \overline{\iota_{\Delta i^*}(\hat{G}_T)}$ was assigned incorrectly in $E_{\iota_{\Delta i^*}(\hat{G}_T)}$, then $\iota_{\Delta i^*}(\hat{G}_T) \in \mathcal{G}_m(\Delta i^*)$ (i.e., in particular $i^* \neq \text{CW}(\iota_{\Delta i^*}(\hat{G}_T))$) would imply $\text{CW}(\mathbf{Q}) \neq i^*$. Consequently, $\mathbf{Q} \in \mathcal{Q}_m^h(i^*)$ lets us infer that $\iota_{\Delta i^*}(\hat{G}_T) \in \mathcal{G}_m(\Delta i^*)$ is only possible if there exists some $\{i,j\} \in \overline{\iota_{\Delta i^*}(\hat{G}_T)}$, which is assigned incorrectly in $E_{\iota_{\Delta i^*}(\hat{G}_T)}$. Regarding that $|\overline{G}| = |E_G|$, we thus get

$$\begin{aligned}&\mathbb{P}_{\mathbf{Q}}\left(\iota_{\Delta i^*}(\hat{G}_T) \in \mathcal{G}_m(\Delta i^*) \text{ and } \overline{\iota_{\Delta i^*}(\hat{G}_T)} = \overline{G}\right) \\ &\leq \mathbb{P}_{\mathbf{Q}}\left(\exists \{i,j\} \in \overline{G}, \text{ which is ass. inc. in } E_{\iota_{\Delta i^*}(\hat{G}_T)} \text{ and } \overline{\iota_{\Delta i^*}(\hat{G}_T)} = \overline{G}\right) \\ &\leq \sum_{\{i,j\} \in \overline{G}} \mathbb{P}_{\mathbf{Q}}\left(\{i,j\} \text{ is ass. inc. in } E_{\iota_{\Delta i^*}(\hat{G}_T)} \text{ and } \overline{\iota_{\Delta i^*}(\hat{G}_T)} = \overline{G}\right) \leq \gamma' |E_G| \mathbb{P}_{\mathbf{Q}}\left(\overline{\iota_{\Delta i^*}(\hat{G}_T)} = \overline{G}\right)\end{aligned}$$

for every $G \in \mathcal{G}_m$. Together with (4) and $\mathbb{P}_{\mathbf{Q}}(|E_{\iota_{\Delta i^*}(\hat{G}_T)}| \leq m) = 1$, which holds according to (b) of Lemma A.5, we infer

$$\begin{aligned}\mathbb{P}_{\mathbf{Q}}(\mathbf{D}(\mathcal{A}) \neq i^*) = \mathbb{P}_{\mathbf{Q}}\left(\iota_{\Delta i^*}(\hat{G}_T) \in \mathcal{G}_m(\Delta i^*)\right) &= \sum_{\overline{G}: G \in \mathcal{G}_m} \mathbb{P}_{\mathbf{Q}}\left(\iota_{\Delta i^*}(\hat{G}_T) \in \mathcal{G}_m(\Delta i^*) \text{ and } \overline{\iota_{\Delta i^*}(\hat{G}_T)} = \overline{G}\right) \\ &\leq \sum_{\overline{G}: G \in \mathcal{G}_m} \mathbb{P}_{\mathbf{Q}}\left(\exists \{i,j\} \in \overline{G}, \text{ which is ass. inc. in } E_{\iota_{\Delta i^*}(\hat{G}_T)} \text{ and } \overline{\iota_{\Delta i^*}(\hat{G}_T)} = \overline{G}\right) \\ &\leq \gamma' \sum_{\overline{G}: G \in \mathcal{G}_m} |\overline{G}| \mathbb{P}_{\mathbf{Q}}\left(\overline{\iota_{\Delta i^*}(\hat{G}_T)} = \overline{G}\right) \leq \gamma' m \leq \alpha,\end{aligned}$$

where we have used that $\sum_{\overline{G}:G\in\mathcal{G}_m} \mathbb{P}_{\mathbf{Q}}(\overline{\mathfrak{l}_{\Delta i^*}(\hat{G}_T)} = \overline{G}) = 1$ holds trivially.

Part 4: Bounding the type II error

Now, we consider the case $\mathbf{Q} \in \mathcal{Q}_m^h(\neg\mathrm{CW})$. Similarly as above in Part 3, Lemma A.5 yields

$$\mathbb{P}_{\mathbf{Q}}(\mathbf{D}(\mathcal{A}) \neq \neg\mathrm{CW}) = \mathbb{P}_{\mathbf{Q}}(\mathbf{D}(\mathcal{A}) \in [m]) = \mathbb{P}_{\mathbf{Q}}(\hat{G}_T \in \mathcal{G}_m(\diamondsuit)). \tag{5}$$

Next, using $\mathfrak{l}_\diamondsuit$ as defined in Lemma A.5, an analogue argumentation as above shows that $\mathfrak{l}_\diamondsuit(\hat{G}_T) \in \mathcal{G}_m(\diamondsuit)$ is only possible if there exists some $\{i,j\} \in \overline{\mathfrak{l}_\diamondsuit(\hat{G}_T)}$, which is assigned incorrectly in $E_{\mathfrak{l}_\diamondsuit(\hat{G}_T)}$. From this and Part 2 we can infer that

$$\mathbb{P}_{\mathbf{Q}}\left(\mathfrak{l}_\diamondsuit(\hat{G}_T) \in \mathcal{G}_m(\diamondsuit) \text{ and } \overline{\mathfrak{l}_\diamondsuit(\hat{G}_T)} = \overline{G}\right) \leq \gamma' |E_G| \mathbb{P}_{\mathbf{Q}}\left(\overline{\mathfrak{l}_{\Delta i^*}(\hat{G}_T)} = \overline{G}\right) \tag{6}$$

is fulfilled for every $G \in \mathcal{G}_m$. According to (b) of Lemma A.5 we have $\mathbb{P}_{\mathbf{Q}}(|E_{\mathfrak{l}_\diamondsuit(\hat{G}_T)}| \leq m-1) = 1$, whence combining (5) with (6) yields

$$\mathbb{P}_{\mathbf{Q}}(\mathbf{D}(\mathcal{A}) \neq \neg\mathrm{CW}) \leq \gamma' \sum_{\overline{G}:G\in\mathcal{G}_m} |\overline{G}| \mathbb{P}_{\mathbf{Q}}\left(\overline{\mathfrak{l}_{\Delta i^*}(\hat{G}_T)} = \overline{G}\right) \leq \gamma'(m-1) \leq \beta.$$

This completes the proof of Part 4 and also the proof of the theorem. □

E DSTAS FOR TESTIFICATION, CHECK_CW AND VERIFY_I_AS_CW

Recall that both testification for the CW (which we may conveniently denote by **Testification** in the following) and **Verify_i_as**_CW can be regarded with and without the $\exists$CW assumption. Thus, for **problem** $\in$ {**Testification**, **Verify_i_as**_CW} we will write **problem**{} and **problem**{$\exists$CW} for the corresponding problem without and with the assumption $\exists$CW, respectively; when we simply write **problem** we mean any of these two variants.

Any DSTA $\mathcal{A}_{\mathrm{Bin}}$ for Testification for the CW of tournaments $G \in \overline{\mathcal{G}}_m$ (recall Fact 5.1) is w.l.o.g. supposed to do in each time step $t \in \mathbb{N}$ *both* of the following steps in succession:

(i) Query one pair $(i,j) \in \langle m \rangle_2$ and receive as feedback 1 if $i \xrightarrow{G} j$ and 0 if $j \xrightarrow{G} i$ holds.
(ii) Either continue (i.e., skip this step) or terminate. In the latter case return some $i \in [m]$ or $\neg$CW.

Similarly, a DSTA $\mathcal{A}_{\mathrm{Bin}}$ for **Check**_CW, resp. **Verify_i_as**_CW(with $i^* \in [m]$ as input), may w.l.o.g. be supposed to do in each time step at first (i) and then

(ii') Either continue (i.e., skip this step) or terminate. In the latter case return CW or $\neg$CW,

resp.

(ii") Either continue (i.e., skip this step) or terminate. In the latter case return i^* or $\neg i^*$.

Let $\mathfrak{A}_m$ (resp. $\mathfrak{A}_m^{\textbf{Check\_}\mathrm{CW}}$ or $\mathfrak{A}_m^{\textbf{Verify\_i\_as\_}\mathrm{CW}}$) denote the set of all DSTAs for testification of the CW (resp. for **Check**_CW or **Verify_i_as**_CW). Recall that DSTAs, which tackle **Verify_i_as**_CW, are given some $i^* \in [m]$ as input; given such a DSTA $\mathcal{A}_{\mathrm{Bin}}$, we will write $\mathcal{A}_{\mathrm{Bin}}[i^*]$ for $\mathcal{A}_{\mathrm{Bin}}$ started with i^*, and in case i^* is fixed or clear from the context we may simply write $\mathcal{A}_{\mathrm{Bin}}$ instead of $\mathcal{A}_{\mathrm{Bin}}[i^*]$ for convenience. With a slight abuse of notation we will call not only $\mathcal{A}_{\mathrm{Bin}}$ but also $\mathcal{A}_{\mathrm{Bin}}[i^*]$ a DSTA for **Verify_i_as**_CW; whenever we simply write a "DSTA for **Verify_i_as**_CW" it should be clear from the context to which of these notions we are referring.

Suppose $\mathcal{A}_{\mathrm{Bin}}$ to be a DSTA for **Testification**, **Check**_CW or **Verify_i_as**_CW, in the latter case of which it is supposed to be started with a fixed $i^* \in [m]$, i.e., $\mathcal{A}_{\mathrm{Bin}} = \mathcal{A}_{\mathrm{Bin}}[i^*]$. As above, we write $T_G^{\mathcal{A}_{\mathrm{Bin}}}$ for the number of queries made by the DSTA $\mathcal{A}_{\mathrm{Bin}}$ started with G until termination. We write for distinct $i,k \in [m]$ that $\mathcal{A}_{\mathrm{Bin}}$ queries $\{i,k\}$ if it queries (i,k) or (k,i), i.e., a query $\{i,k\}$ represents either (i,k) or (k,i). We write $(i_G^{\mathcal{A}_{\mathrm{Bin}}}(t), j_G^{\mathcal{A}_{\mathrm{Bin}}}(t))$ for the query of $\mathcal{A}_{\mathrm{Bin}}$ made at time t when testing G. We say that *i beats j (or equivalently, j is beaten by i) at time t (according to the queries of $\mathcal{A}_{\mathrm{Bin}}$ started with G)* if $\{i_G^{\mathcal{A}_{\mathrm{Bin}}}(t), j_G^{\mathcal{A}_{\mathrm{Bin}}}(t)\} = \{i,j\}$ and $i \xrightarrow{G} j$ hold. We write $T_G^{\mathcal{A}_{\mathrm{Bin}}}$ for the number of queries made by $\mathcal{A}_{\mathrm{Bin}}$ started with G until termination. Then, the worst-case runtime of a DSTA $\mathcal{A}_{\mathrm{Bin}}$, which solves **problem**, is given as

$$T^{\mathcal{A}_{\mathrm{Bin}}} = \begin{cases} \max_{G\in\overline{\mathcal{G}}_m} T_G^{\mathcal{A}_{\mathrm{Bin}}}, & \text{if } \textbf{problem} \in \{\textbf{Testification}\{\}, \textbf{Check\_}\mathrm{CW}\{\}\}, \\ \max_{i^*\in[m]} \max_{G\in\overline{\mathcal{G}}_m} T_G^{\mathcal{A}_{\mathrm{Bin}}[i^*]}, & \text{if } \textbf{problem} = \textbf{Verify\_i\_as\_}\mathrm{CW}\{\}, \\ \max_{G\in\overline{\mathcal{G}}_m(\mathrm{CW})} T_G^{\mathcal{A}_{\mathrm{Bin}}}, & \text{if } \textbf{problem} = \textbf{Check\_}\mathrm{CW}\{\exists\mathrm{CW}\}, \\ \max_{i^*\in[m]} \max_{G\in\overline{\mathcal{G}}_m(\mathrm{CW})} T_G^{\mathcal{A}_{\mathrm{Bin}}[i^*]}, & \text{if } \textbf{problem} = \textbf{Verify\_i\_as\_}\mathrm{CW}\{\exists\mathrm{CW}\}. \end{cases}$$

As querying an already queried pair $\{i, j\}$ for a second time does not provide any more information, we may assume w.l.o.g. that each DSTA $\mathcal{A}_{\text{Bin}}$ started on some $G \in \overline{\mathcal{G}}_m$ queries each pair at most once, i.e., the queries $\{i_G^{\mathcal{A}_{\text{Bin}}}(1), j_G^{\mathcal{A}_{\text{Bin}}}(1)\}, \dots, \{i_G^{\mathcal{A}_{\text{Bin}}}(T_G^{\mathcal{A}_{\text{Bin}}}), j_G^{\mathcal{A}_{\text{Bin}}}(T_G^{\mathcal{A}_{\text{Bin}}})\}$ are pairwise distinct. This implies that $T_G^{\mathcal{A}_{\text{Bin}}} \leq \binom{m}{2}$ for any DSTA $\mathcal{A}_{\text{Bin}}$ and any $G \in \overline{\mathcal{G}}_m$.

In the following, let us write $\mathcal{A}_{\text{Bin}}(G)$ for the output of $\mathcal{A}_{\text{Bin}}$ when started with G.

Recall that a DSTA $\mathcal{A}_{\text{Bin}}$ is *testification$\{\}$-correct* (simply called *testification-correct* in the main paper) if both $\forall G \in \overline{\mathcal{G}}_m(\text{CW}) : \mathcal{A}_{\text{Bin}}(G) = \text{CW}(G)$ and $\forall G \in \overline{\mathcal{G}}_m(\neg\text{CW}) : \mathcal{A}_{\text{Bin}}(G) = \neg\text{CW}$ hold, and that we denoted by $\mathfrak{A}_m^*$ the set of testification-correct algorithms $\mathcal{A}_{\text{Bin}} \in \mathfrak{A}_m$. Similarly, we may say that a DSTA $\mathcal{A}_{\text{Bin}} \in \mathfrak{A}_m^{\textbf{Check\_CW}}$ is *check$\{\}$-correct* if it fulfills

$$\forall G \in \mathcal{G}_m(\text{CW}) : \mathcal{A}_{\text{Bin}}(G) = \text{CW} \quad \text{and} \quad \forall G \in \overline{\mathcal{G}}_m(\neg\text{CW}) : \mathcal{A}_{\text{Bin}}(G) = \neg\text{CW}$$

and $\mathcal{A}_{\text{Bin}} \in \mathfrak{A}_m^{\textbf{Verify\_i\_as\_CW}}$ is called *verify$\{\}$-correct* if

$$\forall i^* \in [m]\, \forall G \in \overline{\mathcal{G}}_m(i^*) : \mathcal{A}_{\text{Bin}}[i^*](G) = i^* \quad \text{and}$$
$$\forall i^* \in [m]\, \forall G \in \overline{\mathcal{G}}_m \setminus \overline{\mathcal{G}}_m(i^*) : \mathcal{A}_{\text{Bin}}[i^*](G) = \neg i^*.$$

Moreover, a DSTA $\mathcal{A}_{\text{Bin}}$ in $\mathfrak{A}_m$ resp. $\mathfrak{A}_m^{\textbf{Verify\_i\_as\_CW}}$ is called *testification$\{\exists\text{CW}\}$-correct* resp. *verification$\{\exists\text{CW}\}$-correct* if it fulfills

$$\forall G \in \overline{\mathcal{G}}_m(\text{CW}) : \mathcal{A}_{\text{Bin}}(G) = \text{CW}(G) \quad \text{and} \quad \forall G \in \overline{\mathcal{G}}_m(\neg\text{CW}) : \mathcal{A}_{\text{Bin}}(G) = \neg\text{CW}$$

resp.

$$\forall i^* \in [m]\, \forall G \in \overline{\mathcal{G}}_m(i^*) : \mathcal{A}_{\text{Bin}}[i^*](G) = i^* \quad \text{and}$$
$$\forall i^* \in [m]\, \forall G \in \overline{\mathcal{G}}_m(\text{CW}) \setminus \overline{\mathcal{G}}_m(i^*) : \mathcal{A}_{\text{Bin}}[i^*](G) = \neg i^*.$$

For $G \in \mathcal{G}_m$ with $(i, j) \in E_G$ or $(j, i) \in E_G$ we define for convenience

$$\{i, j\}_G := \begin{cases} (i, j), & \text{if } (i, j) \in E_G, \\ (j, i), & \text{if } (j, i) \in E_G. \end{cases}$$

As every DSTA $\mathcal{A}_{\text{Bin}}$ is deterministic (i.e., their decisions in (i) and (ii) are not random), we have the following: For every $G, G' \in \overline{\mathcal{G}}_m$ we have then $(i_G^{\mathcal{A}_{\text{Bin}}}(1), j_G^{\mathcal{A}_{\text{Bin}}}(1)) = (i_{G'}^{\mathcal{A}_{\text{Bin}}}(1), j_{G'}^{\mathcal{A}_{\text{Bin}}}(1))$. Moreover, if $t_0 \leq T_G^{\mathcal{A}_{\text{Bin}}}$ is such that $\{i_G^{\mathcal{A}_{\text{Bin}}}(t), j_G^{\mathcal{A}_{\text{Bin}}}(t)\}_G = \{i_G^{\mathcal{A}_{\text{Bin}}}(t), j_G^{\mathcal{A}_{\text{Bin}}}(t)\}_{G'}$ holds for all $t \leq t_0$, then $(i_G^{\mathcal{A}_{\text{Bin}}}(t+1), j_G^{\mathcal{A}_{\text{Bin}}}(t+1)) = (i_{G'}^{\mathcal{A}_{\text{Bin}}}(t+1), j_{G'}^{\mathcal{A}_{\text{Bin}}}(t+1))$ is fulfilled.

Moreover, we define $\mathfrak{G}_G^{\mathcal{A}_{\text{Bin}}}(t)$ to be the "picture" which algorithm $\mathcal{A}_{\text{Bin}}$, started with G, has of G at time t. More precisely, $\mathfrak{G}_G^{\mathcal{A}_{\text{Bin}}}(t) = ([m], E_{\mathfrak{G}_G^{\mathcal{A}_{\text{Bin}}}(t)})$ is given via

$$E_{\mathfrak{G}_G^{\mathcal{A}_{\text{Bin}}}(t)} = \bigcup\nolimits_{t' \leq t-1} \left\{ \{i_G^{\mathcal{A}_{\text{Bin}}}(t'), j_G^{\mathcal{A}_{\text{Bin}}}(t')\}_G \right\}.$$

In particular, $\mathfrak{G}_G^{\mathcal{A}_{\text{Bin}}}(t)$ is a subgraph of G in the sense that $E_{\mathfrak{G}_G^{\mathcal{A}_{\text{Bin}}}(t)} \subseteq E_G$.

The following lemma is crucial for exploiting DSTAs in the noisy tournament sampling setting. It gives necessary conditions on the observations, which have to be made by any correct DSTA until termination.

Lemma E.1. *(i) Let $\mathcal{A}_{\text{Bin}}$ be a DSTA and $G \in \overline{\mathcal{G}}_m$. If $\mathcal{A}_{\text{Bin}}$ is testification$\{\}$-, check- or verify$\{\}$-correct (and given input $i^* \in [m]$ in the latter case), then $\mathfrak{G}_G^{\mathcal{A}_{\text{Bin}}}(T_G^{\mathcal{A}_{\text{Bin}}})$ is an element of*

$$\mathcal{G}_m(\neg\text{CW}) \cup \bigcup\nolimits_{i^* \in [m]} \mathcal{G}_m(i^*), \quad \mathcal{G}_m(\text{CW}) \cup \mathcal{G}_m(\neg\text{CW}), \quad \textit{or} \quad \mathcal{G}_m(i^*) \cup \mathcal{G}_m(\neg i^*),$$

respectively.

(ii) Let $\mathcal{A}_{\text{Bin}}$ be a DSTA and $G \in \overline{\mathcal{G}}_m(\text{CW})$. If $\mathcal{A}_{\text{Bin}}$ is testification$\{\exists\text{CW}\}$- or verify$\{\exists\text{CW}\}$-correct (and given input $i^* \in [m]$ in the latter case), then $\mathfrak{G}_G^{\mathcal{A}_{\text{Bin}}}(T_G^{\mathcal{A}_{\text{Bin}}})$ is an element of

$$\bigcup\nolimits_{i^* \in [m]} \mathcal{G}_m(i^*) \quad \textit{or} \quad \mathcal{G}_m(i^*) \cup \mathcal{G}_m(\neg i^*),$$

respectively.

Proof of Lemma E.1. (i) Suppose $\mathcal{A}_{\mathrm{Bin}}$ to be testification$\{\}$-correct, let $G \in \overline{\mathcal{G}}_m$ be fixed and assume $\mathfrak{G}_G^{\mathcal{A}_{\mathrm{Bin}}}(T_G^{\mathcal{A}_{\mathrm{Bin}}}) \notin \mathcal{G}_m(\mathrm{CW}) \cup \bigcup_{i^* \in [m]} \mathcal{G}_m(i^*)$. Then, there exist two extensions $G_0, G_1 \in \overline{\mathcal{G}}_m$ of G with $G_b \in \overline{\mathcal{G}}_m(X_b)$, $b \in \{0,1\}$, wherein X_0 and X_1 are distinct elements of $\{\neg\mathrm{CW}, 1, \ldots, m\}$. As both G_0 and G_1 are extensions of $\mathfrak{G}_G^{\mathcal{A}_{\mathrm{Bin}}}(T_G^{\mathcal{A}_{\mathrm{Bin}}})$, we have $T_G^{\mathcal{A}_{\mathrm{Bin}}} = T_{G_b}^{\mathcal{A}_{\mathrm{Bin}}}$, $\mathfrak{G}_G^{\mathcal{A}_{\mathrm{Bin}}}(T_G^{\mathcal{A}_{\mathrm{Bin}}}) = \mathfrak{G}_{G_b}^{\mathcal{A}_{\mathrm{Bin}}}(T_{G_b}^{\mathcal{A}_{\mathrm{Bin}}})$ and in particular $\mathcal{A}_{\mathrm{Bin}}(G_b) = \mathcal{A}_{\mathrm{Bin}}(G)$ for every $b \in \{0,1\}$, i.e., $\mathcal{A}_{\mathrm{Bin}}$ either classifies G_0 or G_1 incorrectly. This contradicts the assumption that $\mathcal{A}_{\mathrm{Bin}}$ is testification$\{\}$-correct, and the first statement follows. The results for check- and verify$\{\}$-correct algorithms can be shown similarly.

(ii) Now, suppose $\mathcal{A}_{\mathrm{Bin}}$ to be testification$\{\exists\mathrm{CW}\}$-correct, let $G \in \overline{\mathcal{G}}_m(\mathrm{CW})$ and assume $\mathfrak{G}_G^{\mathcal{A}_{\mathrm{Bin}}}(T_G^{\mathcal{A}_{\mathrm{Bin}}}) \notin \bigcup_{i^* \in [m]} \mathcal{G}_m(i^*)$. Similarly as in (ii) there exist two extensions $G_0, G_1 \in \overline{\mathcal{G}}_m$ of G with $G_b \in \overline{\mathcal{G}}_m(X_b)$, $b \in \{0,1\}$, where X_0 and X_1 are distinct elements of $\{1, \ldots, m\}$. As above, we infer that $\mathcal{A}$ classifies either G_0 or G_1 incorrectly, in contradiction to testification$\{\exists\mathrm{CW}\}$-correctness of $\mathcal{A}_{\mathrm{Bin}}$. This shows the first statement, while the second one can be seen analogously.

□

Next, let us provide the upper and lower bounds for the worst-case runtimes of correct DSTAs for **Testification**, **Check**_CW and **Verify_i_as**_CW as stated in Table 1.

• **Verify_i_as**_$\mathrm{CW}\{\}$ **and Verify_i_as**_$\mathrm{CW}\{\exists\mathrm{CW}\}$**:** It is easy to check that Algorithm 2 is a correct solution to **Verify_i_as**_$\mathrm{CW}\{\}$ and makes exactly $m-1$ queries. It is also a solution to the less difficult task **Verify_i_as**_$\mathrm{CW}\{\exists\mathrm{CW}\}$, i.e., solving **Verify_i_as**_$\mathrm{CW}\{\exists\mathrm{CW}\}$ is also feasible with worst-case running time $m-1$.

To see that $m-1$ queries are not only sufficient but also necessary for **Verify_i_as**_$\mathrm{CW}\{\exists\mathrm{CW}\}$, suppose $\mathcal{A}_{\mathrm{Bin}}$ to be a verify$\{\exists\mathrm{CW}\}$-correct DSTA. Let $i \in [m]$ and $G \in \overline{\mathcal{G}}_m(\mathrm{CW})$ with $\mathrm{CW}(G) = i$. Assuming $\mathcal{A}_{\mathrm{Bin}}[i]$ does not query $\{i,j\}$ for some $j \in [m] \setminus \{i\}$ before termination would imply that there exist extensions $G_0, G_1 \in \overline{\mathcal{G}}_m$ of $\mathfrak{G}_G^{\mathcal{A}_{\mathrm{Bin}}}(T_G^{\mathcal{A}_{\mathrm{Bin}}[i]})$ such that $i = \mathrm{CW}(G_0)$ and $i \neq \mathrm{CW}(G_1)$ hold. Hence, $\mathcal{A}_{\mathrm{Bin}}[i]$ would classify either G_0 or G_1 incorrectly, a contradiction. This shows, that $\mathcal{A}_{\mathrm{Bin}}[i]$ has to make all the $m-1$ queries $\{i,j\}$, $j \neq i$. As the problem **Verify_i_as**_$\mathrm{CW}\{\}$ is more complex[4] than **Verify_i_as**_$\mathrm{CW}\{\exists\mathrm{CW}\}$, this shows also that solving **Verify_i_as**_$\mathrm{CW}\{\}$ requires a worst-case running time of $m-1$.

Algorithm 2 An optimal DSTA for **Verify_i_as**_$\mathrm{CW}\{\}$ (and also for **Verify_i_as**_$\mathrm{CW}\{\exists\mathrm{CW}\}$)

Input: $i \in [m]$
Initialization: $W \leftarrow [m] \setminus \{i\}$
▷ W = set of nodes $j \in [m]$, which have not yet been compared to i

```
while |W| ≥ 1 do
    Choose an arbitrary j ∈ W
    (i', j') ← {i, j}_G                      ▷ Query {i, j}
    if (i, j') = (j, i) then return ¬i        ▷ i can not be the CW
    else W ← W \ {j}                          ▷ j has been compared to i
return i
```

• **Testification**$\{\}$**:**

Proposition E.2. *Algorithm 3, which we denote by* $\mathcal{A}^*_{\mathrm{Bin}}$*, is testification*$\{\}$*-correct and fulfills* $T^{\mathcal{A}^*_{\mathrm{Bin}}} = 2m - \lfloor \log m \rfloor - 2$*. Moreover, if a DSTA* $\mathcal{A}_{\mathrm{Bin}}$ *is testification*$\{\}$*-correct or check-correct, then* $T^{\mathcal{A}_{\mathrm{Bin}}} \geq 2m - \lfloor \log m \rfloor - 2$*.*

Proof. Confer Theorem 2.1 in [Procaccia, 2008] and Lemma 3.2 in [Balasubramanian et al., 1997]. □

[4]Note that every solution $\mathcal{A}_{\mathrm{Bin}}$ for **Verify_i_as**_$\mathrm{CW}\{\}$ is also a solution to **Verify_i_as**_$\mathrm{CW}\{\exists\mathrm{CW}\}$.

Algorithm 3 An optimal DSTA for Testification$\{\}$[Procaccia, 2008]

Initialization: Construct an *almost complete* binary tree T of height $D := \lceil \log m \rceil$ with m leaves, which are labeled by $1, \ldots, m$. Here, *almost complete* means that there are exactly 2^d nodes on each level $d \leq D - 1$.

while $\text{height}(T) > 0$ **do**
  Pick two sibling leave nodes $i, j \in [m]$ of T and compare them
  if $\{i, j\}_G = (i, j)$ **then** ▷ j can not be the CW
    Label the unique parent of i and j with i, then remove its children from T
  else ▷ i can not be the CW
    Label the unique parent of i and j with j, then remove its children from T
Let i^* be the label of the only node in T
Compare i^* with all other alternatives, with which it has *not* been compared yet
if i^* has won all of its duels **then return** i^*
else return $\neg\mathrm{CW}$

• **Testification**$\{\exists\mathrm{CW}\}$**:** Suppose $\mathcal{A}_{\mathrm{Bin}}$ to be testification$\{\exists\mathrm{CW}\}$-correct and define an algorithm $\mathcal{A}'_{\mathrm{Bin}}$ for **Verify_i_as_**CW as follows: Given some $i \in [m]$, simulate $\mathcal{A}_{\mathrm{Bin}}$ and return CW if $\mathcal{A}_{\mathrm{Bin}}$ outputs i, and return $\neg i$ otherwise. Then, $\mathcal{A}_{\mathrm{Bin}}$ is correct iff $\mathcal{A}'_{\mathrm{Bin}}$ is correct. As $\mathcal{A}_{\mathrm{Bin}}$ and $\mathcal{A}'_{\mathrm{Bin}}$ have exactly the same sample complexity – regardless of the input i to $\mathcal{A}'_{\mathrm{Bin}}$, we obtain with regard to the lower bound on the worst-case running time of correct algorithms to **Verify_i_as_**CW$\{\exists\mathrm{CW}\}$ that $T^{\mathcal{A}_{\mathrm{Bin}}} = \max_{G \in \overline{\mathcal{G}}_m(\mathrm{CW})} T_G^{\mathcal{A}_{\mathrm{Bin}}} \geq \max_{G \in \overline{\mathrm{CW}}, i \in [m]} T_G^{\mathcal{A}'_{\mathrm{Bin}}[i]} \geq m - 1$.

To see the upper bound, consider Algorithm 4, which we denote by $\mathcal{A}_{\mathrm{Bin}}$ for the moment. Therein, $|S|$ decreases by 1 in each iteration of the while loop, whence $\mathcal{A}_{\mathrm{Bin}}$ terminates after exactly $m - 1$ time steps when started on any $G \in \overline{\mathcal{G}}_m$. In particular, $T^{\mathcal{A}_{\mathrm{Bin}}} \leq m - 1$ holds. For $G \in \overline{\mathcal{G}}_m(\mathrm{CW})$ this algorithm returns at termination the only remaining element i of S. The construction of S assures $\mathrm{CW}(G) \neq j$ for each $j \in [m] \setminus S$, i.e., $\mathcal{A}_{\mathrm{Bin}}(G) = \mathrm{CW}(G)$ has to be fulfilled. This shows, that $\mathcal{A}_{\mathrm{Bin}}$ is a testification$\{\exists\mathrm{CW}\}$-correct.

Algorithm 4 An optimal DSTA for Testification$\{\exists\mathrm{CW}\}$

Initialization: $S \leftarrow [m]$, $i \leftarrow 1$ ▷ $S =$ set of candidates for CW
▷ $i =$ the current candidate

while $|S| > 1$ **do**
  Choose an arbitrary $j \in S \setminus \{i\}$
  $(i', j') \leftarrow \{i, j\}_G$ ▷ Query $\{i, j\}$
  if $(i, j') = (i, j)$ **then** $S \leftarrow S \setminus \{j\}$ ▷ j can not be the CW
  else $S \leftarrow S \setminus \{i\}$, $i \leftarrow j$ ▷ i is not the CW, j is the new candidate
return i ▷ $S = \{i\}$

• **Check_**CW**:** Let $\mathcal{A}'_{\mathrm{Bin}}$ be the following DSTA: Started on $G \in \overline{\mathcal{G}}_m$, it simulates Algorithm 3, denoted by $\mathcal{A}_{\mathrm{Bin}}$, on G until it terminates and then chooses as output $\neg\mathrm{CW}$ if $\mathcal{A}_{\mathrm{Bin}}(G) = \neg\mathrm{CW}$, and CW otherwise. As $\mathcal{A}_{\mathrm{Bin}}$ is testification$\{\}$-correct, $\mathcal{A}'_{\mathrm{Bin}}$ is check$\{\}$-correct. Moreover, $\mathcal{A}'_{\mathrm{Bin}}$ needs exactly as many samples as $\mathcal{A}_{\mathrm{Bin}}$, so that we have $T^{\mathcal{A}'_{\mathrm{Bin}}} \leq 2m - \lfloor \log m \rfloor - 2$.

As **Check_**CW is easier than Testification$\{\}$, Proposition E.2 yields that $\mathcal{A}'_{\mathrm{Bin}}$ is optimal w.r.t. the worst-case runtime.

F RESULTS FOR CHECK_CW, FIND_CW AND VERIFY_I_AS_CW IN THE PROBABILISTIC SETTING

As in Section E, for **problem** $\in$ $\{$**Testification, Verify_i_as_**CW$\}$, we write **problem**$\{\}$ and **problem**$\{\exists\mathrm{CW}\}$ for the corresponding problem without and with the assumption $\exists\mathrm{CW}$, respectively; when we simply write **problem** we mean any of these two variants. If $\mathcal{A}$ tackles **Verify_i_as_**CW, we write $\mathcal{A}[i]$ for $\mathcal{A}$ with input i.

For given error probabilities $\alpha, \beta \in (0, 1)$, we say that an algorithm $\mathcal{A}$ *solves* ***Check_***CW on $\mathcal{Q}_m^h$ if $T^{\mathcal{A}}$ is almost surely finite and the following holds:

$$\inf_{\mathbf{Q}\in\mathcal{Q}_m^h(\mathrm{CW})} \mathbb{P}_{\mathbf{Q}}(\mathbf{D}(\mathcal{A}) = \mathrm{CW}) \geq 1-\alpha \quad \text{and} \\ \inf_{\mathbf{Q}\in\mathcal{Q}_m^h(\neg\mathrm{CW})} \mathbb{P}_{\mathbf{Q}}(\mathbf{D}(\mathcal{A}) = \neg\,\mathrm{CW}) \geq 1-\beta.$$

Similarly, we say $\mathcal{A}$ *solves* ***Verify_i_as_***$\mathrm{CW}\{\}$ on $\mathcal{Q}_m^h$ if, given any $i \in [m]$, $T^{\mathcal{A}[i]}$ is almost surely finite and the following holds:

$$\inf_{\mathbf{Q}\in\mathcal{Q}_m^h(\mathrm{CW}):\mathrm{CW}(\mathbf{Q})=i} \mathbb{P}_{\mathbf{Q}}(\mathbf{D}(\mathcal{A}[i]) = i) \geq 1-\alpha \quad \text{and} \\ \inf_{\mathbf{Q}\in\mathcal{Q}_m^h:\mathbf{Q}\in\mathcal{Q}_m(\neg\mathrm{CW}) \text{ or } \mathrm{CW}(\mathbf{Q})\neq i} \mathbb{P}_{\mathbf{Q}}(\mathbf{D}(\mathcal{A}[i]) = \neg i) \geq 1-\beta. \tag{7}$$

The notions of *a solution* to **Testification**$\{\exists\mathrm{CW}\}$ resp. **Verify_i_as_**$\mathrm{CW}\{\exists\mathrm{CW}\}$ is defined accordingly by considering relations $\mathbf{Q}$ in $\mathcal{Q}_m^h(\mathrm{CW})$ (instead of only $\mathcal{Q}_m^h$) for the desired theoretical guarantees.

In the following, we will show the following lower and upper bounds for worst-case expected running times on $\mathcal{Q}_m^h$ for solutions to the problems **Testification** and **Verify_i_as_**CW under no assumption. The lower and upper bounds for **Testification**$\{\exists\mathrm{CW}\}$, which is also known as the *best-arm-identification* problem, have already been shown in [Braverman et al., 2016]. The upper bound for **Verify_i_as_**$\mathrm{CW}\{\exists\mathrm{CW}\}$ is a direct consequence of the upper bound for **Verify_i_as_**$\mathrm{CW}\{\}$.

	no assumption	assumption: $\exists\mathrm{CW}$
Check_CW	$\mathcal{O}(m\ln(m)h^{-2}\ln(\gamma^{-1}))$ $\Omega(mh^{-2}\ln\gamma^{-1})$	–
Verify_i_as_CW	$\mathcal{O}(m\ln(m)h^{-2}\ln(\gamma^{-1}))$ $\Omega(mh^{-2}\ln\gamma^{-1})$	$\mathcal{O}(m\ln(m)h^{-2}\ln(\gamma^{-1}))$ $\Omega(mh^{-2}\ln\gamma^{-1})$
Testification	$\mathcal{O}(m\ln(m)h^{-2}\ln(\gamma^{-1}))$ $\Omega(mh^{-2}\ln\gamma^{-1})$	$\widetilde{\Theta}(mh^{-2}\ln\gamma^{-1})$

In order to see the upper bound for **Check_**CW and **Verify_i_as_**$\mathrm{CW}\{\}$ in the probabilistic setting suppose $h \in (0, 1/2)$, $\gamma_0 \in (0,1)$ and $\alpha, \beta \in (0, \gamma_0)$ to be fixed. Moreover, denote by $\mathcal{A}^{\mathrm{NTS}}$ the corresponding algorithm from Corollary 5.4 called with parameters h, α and β. Now, let $\widehat{\mathcal{A}}_1$ be the algorithm, which simulates $\mathcal{A}^{\mathrm{NTS}}$, terminates as soon as $\mathcal{A}^{\mathrm{NTS}}$ terminates, and outputs

$$\mathbf{D}(\widehat{\mathcal{A}}_1) = \begin{cases} \neg\mathrm{CW}, & \text{if } \mathbf{D}(\mathcal{A}^{\mathrm{NTS}}) = \neg\mathrm{CW}, \\ \mathrm{CW}, & \text{if } \mathbf{D}(\mathcal{A}^{\mathrm{NTS}}) = i \text{ for some } i \in [m]. \end{cases}$$

Similarly, define $\widehat{\mathcal{A}}_2$ to be the algorithm, which, given any $i \in [m]$, simulates $\mathcal{A}^{\mathrm{NTS}}$, terminates as soon as $\mathcal{A}^{\mathrm{NTS}}$ terminates and then returns

$$\mathbf{D}(\widehat{\mathcal{A}}_2[i]) = \begin{cases} i, & \text{if } \mathbf{D}(\mathcal{A}^{\mathrm{NTS}}) = i, \\ \neg i, & \text{otherwise.} \end{cases}$$

Since $\mathcal{A}^{\mathrm{NTS}}$ solves **Testification**$\{\}$ on $\mathcal{Q}_m^h$ for α, β, it follows that $\widehat{\mathcal{A}}_1$ resp. $\widehat{\mathcal{A}}_2$ solves **Check_**CW resp. **Verify_i_as_**$\mathrm{CW}\{\}$ on $\mathcal{Q}_m^h$ for α, β. Moreover, both $\widehat{\mathcal{A}}_1$ and $\widehat{\mathcal{A}}_2$ have exactly the same runtime as $\mathcal{A}^{\mathrm{NTS}}$. Consequently, we have with regard to Corollary 5.4

$$\sup_{\mathbf{Q}\in\mathcal{Q}_m^h} \mathbb{E}_{\mathbf{Q}}[T^{\widehat{\mathcal{A}}_1}] \in \mathcal{O}\left(m\ln(m)h^{-2}\ln(\gamma^{-1})\right)$$

and also

$$\sup_{\mathbf{Q}\in\mathcal{Q}_m^h} \max_{i\in[m]} \mathbb{E}_{\mathbf{Q}}[T^{\widehat{\mathcal{A}}_2[i]}] \in \mathcal{O}\left(m\ln(m)h^{-2}\ln(\gamma^{-1})\right)$$

as $\max\{m, h^{-1}, \gamma^{-1}\} \to \infty$, respectively.

We prepare our instance-wise lower bounds on solutions to **Check_**CW and **Verify_i_as_**$\mathrm{CW}\{\}$ with the following Lemma.

Lemma F.1. *Let $\mathbf{Q} \in \mathcal{Q}_m$ be such that $q_{i,j} \neq 1/2$ for every distinct i, j in $[m]$. Then, there exists a permuation σ on $[m]$ such that $q_{\sigma(i),\sigma(i+1)} > 1/2$ for every $i \in [m-1]$.*

Proof. Suppose without loss of generality that $q_{i,j} \in \{0,1\}$ for every $(i,j) \in (m)_2$ and recall the bijection Φ from Fact 5.1. By the Theorem of Rédei (cf. [Sachs, 1971]), $G := ([m], E_G) := \Phi(\mathbf{Q})$ contains a Hamiltonian path, i.e., there exists a permutation σ on $[m]$ such that $(\sigma(i), \sigma(i+1)) \in E_G$ for every $i \in [m-1]$. Regarding the definition of Φ, this proves the statement. □

With this, we can state the following lower bounds for **Check**_CW and **Verify_i_as**_CW{}, whose proofs are deferred to Section G.

Theorem F.2. *For any fixed $h_0, \gamma_0 \in (0, 1/2)$ there exists a constant $c = c(h_0, \gamma_0) > 0$ such that the following holds:*

*(i) Let $h \in (0, h_0)$, $\alpha, \beta \in (0, \gamma_0)$ and suppose that $\mathcal{A}$ is some (probabilistic) algorithm, which solves **Verify_i_as**_CW{∃CW} (or the more complex problem **Verify_i_as**_CW{}) on $\mathcal{Q}_m^h$ for α and β. Moreover, let $\mathbf{Q} \in \mathcal{Q}_m^h$ be fixed, define $h_{i,j} = |q_{i,j} - 1/2|$ for every i, j. Then, for all $i \in [m]$ and every $\mathbf{Q} \in \mathcal{Q}_m^h(\mathrm{CW})$ with $\mathrm{CW}(\mathbf{Q}) = i$, we have*

$$\mathbb{E}_{\mathbf{Q}} \left[T^{\mathcal{A}[i]} \right] \geq c \sum\nolimits_{j \in [m] \setminus \{i\}} h_{i,j}^{-2} \ln((\max\{\alpha, \beta\})^{-1}).$$

In particular, $\mathcal{A}$ fulfills for each $i \in [m]$ the estimate

$$\sup\nolimits_{\mathbf{Q} \in \mathcal{Q}_m^h} \mathbb{E}_{\mathbf{Q}}[T^{\mathcal{A}[i]}] \geq \frac{c(m-1) \ln\left((\max\{\alpha, \beta\})^{-1}\right)}{h^2}.$$

*(ii) Let $h \in (0, h_0)$, $\alpha, \beta \in (0, \gamma_0)$ and suppose that $\mathcal{A}$ is some (probabilistic) algorithm, which solves **Check**_CW on $\mathcal{Q}_m^h$ for α, β. Moreover, let $\mathbf{Q} \in \mathcal{Q}_m^h(\mathrm{CW})$ be fixed, define $h_{i,j} = |q_{i,j} - 1/2|$ for every i, j and suppose σ to be a permutation[5] on $[m]$ such that $q_{\sigma(i), \sigma(j)} > 1/2$ iff $i < j$. Then,*

$$\mathbb{E}_{\mathbf{Q}} \left[T^{\mathcal{A}} \right] \geq c \sum\nolimits_{j=3}^{m} h_{\sigma(1), \sigma(j)}^{-2} \ln(\gamma^{-1}) \geq c \min\nolimits_{j \in [m] \setminus \{\mathrm{CW}(\mathbf{Q})\}} \sum\nolimits_{j' \in [m] \setminus \{\mathrm{CW}(\mathbf{Q}), j\}} h_{\mathrm{CW}(\mathbf{Q}), j'}^{-2} \ln(\gamma^{-1}).$$

In particular, $\mathcal{A}$ fulfills

$$\sup\nolimits_{\mathbf{Q} \in \mathcal{Q}_m^h} \mathbb{E}_{\mathbf{Q}}[T^{\mathcal{A}}] \geq \frac{c(m-2) \ln\left((\max\{\alpha, \beta\})^{-1}\right)}{h^2}.$$

We conclude this section with an enhanced algorithm for **Verify_i_as**_CW.

Algorithm 5 $\mathcal{A}^{\mathrm{NTS}}_{\textbf{Verify\_i\_as}\_\mathrm{CW}\{\}}$: Noisy tournament sampling for **Verify_i_as**_CW

Input: $\alpha, \beta, h, \pi, i^*$
Initialization: $\boldsymbol{n}_0, \boldsymbol{w}_0 \leftarrow (0)_{1 \leq i,j \leq m}$, $\hat{E}_0 \leftarrow \emptyset$, $\gamma' \leftarrow \min\{\alpha, \frac{\beta}{m-1}\}$, $C_{h,\gamma'}$ as in (7)

1: **for** $t \in \mathbb{N}$ **do**
2: Do steps 2–11 of Algorithm 1
3: **if** $\hat{G}_t \in \mathcal{G}_m(i^*)$ **then return** i^*
4: **if** $\hat{G}_t \in \mathcal{G}_m(\neg i^*)$ **then return** $\neg i^*$

Theorem F.3. *Let $\pi \in \Pi_\infty$, $h \in (0, 1/2)$ and $\alpha, \beta \in (0, 1)$ be fixed. Then, $\mathcal{A}^{\mathrm{NTS}}_{\textbf{Verify\_i\_as}\_\mathrm{CW}\{\}}$ (Algorithm 5) called with the parameters h, α, β, $\boldsymbol{n}_0 = \mathbf{0} = \boldsymbol{w}_0$ and π as the sampling strategy, solves **Verify_i_as**_CW{} on $\mathcal{Q}_m^h$ for α and β.*

Proof. Let us abbreviate $\mathcal{A} := \mathcal{A}^{\mathrm{NTS}}_{\textbf{Verify\_i\_as}\_\mathrm{CW}\{\}}$ and write $T := T^{\mathcal{A}}$, and suppose $\mathcal{A}$ is given a fixed $i^* \in [m]$ as input. For the sake of convenience we simply write $\mathcal{A}$ for $\mathcal{A}[i^*]$ in the following. Let $\mathbf{Q} \in \mathcal{Q}_m^h$ be arbitrary but fixed. Due to $\overline{\mathcal{G}}_m(i^*) \cup \overline{\mathcal{G}}_m(\neg i^*)$, we infer similarly as in the proof of Theorem 5.2 that $\mathcal{A}$ terminates almost surely. Moreover, using the notation from the proof of Theorem 5.2 we obtain also

$$\mathbb{P}_{\mathbf{Q}}(\{i,j\} \text{ is assigned incorrectly in } \hat{E}_T) \leq \gamma'$$

for every $\{i, j\} \in [m]_2$ with $|q_{i,j} - 1/2| > h$.

[5] For the existence of such a permutation confer Lemma F.1

To bound the type I error, suppose that $\mathbf{Q} \in \mathcal{Q}_m^h(i^*)$. Then, Lemma A.5 ensures

$$\mathbb{P}_{\mathbf{Q}}(\mathbf{D}(\mathcal{A}) \neq i^*) = \mathbb{P}_{\mathbf{Q}}(\mathbf{D}(\mathcal{A}) = \neg i^*) = \mathbb{P}_{\mathbf{Q}}(\mathfrak{l}_{\neg i^*}(\hat{G}_T) \in \mathcal{G}_m(\neg i^*)). \tag{8}$$

The same argumentation as in the proof of Theorem 5.2 yields

$$\mathbb{P}_{\mathbf{Q}}\left(\mathfrak{l}_{\neg i^*}(\hat{G}_T) \in \mathcal{G}_m(\neg i^*) \text{ and } \overline{\mathfrak{l}_{\neg i^*}(\hat{G}_T)} = \overline{G}\right) \leq \gamma' |E_G| \mathbb{P}_{\mathbf{Q}}\left(\overline{\mathfrak{l}_{\neg i^*}(\hat{G}_T)} = \overline{G}\right)$$

for every $G \in \mathcal{G}_m$. Combining this with (8), using $\sum_{\overline{G}:G\in\mathcal{G}_m} \mathbb{P}_{\mathbf{Q}}\left(\overline{\mathfrak{l}_{\neg i^*}(\hat{G}_T)} = \overline{G}\right) \leq 1$ and the fact that $|E_{\mathfrak{l}_{\neg i^*}(\hat{G}_T)}| = 1$ holds a.s. (see (b) of Lemma A.5) shows that

$$\mathbb{P}_{\mathbf{Q}}(\mathbf{D}(\mathcal{A}) \neq i^*) = \sum_{\overline{G}:G\in\mathcal{G}_m} \mathbb{P}_{\mathbf{Q}}\left(\mathfrak{l}_{\neg i^*}(\hat{G}_T) \in \mathcal{G}_m(\neg i^*) \text{ and } \overline{\mathfrak{l}_{\neg i^*}(\hat{G}_T)} = \overline{G}\right) \leq \gamma'.$$

For showing the guarantee on the type II error, let $\mathbf{Q} \in \mathcal{Q}_m^h \setminus \mathcal{Q}_m^h(i^*)$ be fixed. Again, a similar argumentation as in the proof of Theorem 5.2 yields

$$\begin{aligned}\mathbb{P}_{\mathbf{Q}}(\mathbf{D}(\mathcal{A}) \neq \neg i^*) &= \mathbb{P}_{\mathbf{Q}}(\mathbf{D}(\mathcal{A}) = i^*) = \mathbb{P}_{\mathbf{Q}}(\hat{G}_T \in \mathcal{G}_m(i^*)) = \mathbb{P}_{\mathbf{Q}}(\mathfrak{l}_{i^*}(\hat{G}_T) \in \mathcal{G}_m(i^*)) \\ &\leq \gamma' \sum_{\overline{G}:G\in\mathcal{G}_m} |\overline{G}| \mathbb{P}_{\mathbf{Q}}(\mathfrak{l}_{i^*}(\hat{G}_T) = G) \leq (m-1)\gamma' \leq \beta,\end{aligned}$$

where we have used that $|E_{\mathfrak{l}_{i^*}(\hat{G}_T)}| \leq m-1$ holds a.s. according to (b) of Lemma A.5. □

G PROOFS OF THEOREM 4.1 AND THEOREM F.2

This section is dedicated to prove the lower bounds stated in Theorem 4.1 and Theorem F.2. For this purpose, we reduce the problem of testing the following hypothesis to the problems of interest. Let J be some finite index set and suppose we are given independent coins $C_j, j \in J$, with unknown head probabilities $p_j, j \in J$, respectively. For fixed (unknown) $\boldsymbol{p} = (p_j)_{j\in J}$, throwing coin C_j at time t results in the feedback $Y_t^{[p_j]} \sim \mathrm{Ber}(p_j)$, and we suppose the feedback is independent over time and coins, i.e., $\{Y_t^{[p_j]}\}_{j\in J, t\in\mathbb{N}}$ is independent. Let us define the hypothesis

$$\mathbf{H}_{0;J} : \forall j \in J : p_j \geq \frac{1}{2} \qquad \text{and} \qquad \mathbf{H}_{1;J} : \exists j \in J : p_j < \frac{1}{2}.$$

If $\mathcal{A}$ is a (sequential probabilistic) testing algorithm for $\mathbf{H}_{0;J}$ versus $\mathbf{H}_{1;J}$, we may write $\mathbf{D}(\mathcal{A}) = 0$ if $\mathcal{A}$ decides for $\mathbf{H}_{0;J}$ (or rather does not reject), and $\mathbf{D}(\mathcal{A}) = 1$ otherwise. Moreover, write $T^{\mathcal{A}}$ for the stopping time of the algorithm, that is, the number of samples queried until termination, i.e., the total number of coin tosses until termination. We denote by $\mathbb{P}_{\boldsymbol{p}}$ the probability distribution on the different possible states of the algorithm, if the true parameter is $\boldsymbol{p}$, and write $\mathbb{E}_{\boldsymbol{p}}$ for the expectation with respect to $\mathbb{P}_{\boldsymbol{p}}$. For the proofs of Theorem 4.1 and Theorem F.2 we will make use of the following lower bound on the expected termination time $\mathbb{E}_{\boldsymbol{p}}[T^{\mathcal{A}}]$ for algorithms $\mathcal{A}$ which test the above mentioned hypothesis[6] , if it is known in advance that $|p_j - \frac{1}{2}| = h_j$ holds for every $j \in J$ and some $\{h_j\}_{j\in J} \subseteq (0, 1/2)$. The result stated is a consequence of Lemma B.1 and thus relies on the optimality of the sequential probability ratio test. For the sake of completeness, we provide a detailed proof.

Lemma G.1. *Let $h_0, \gamma_0 \in (0, 1/2)$ be fixed, $\gamma \in (0, \gamma_0)$ and J be some arbitrary finite index set. For $h \in (0, h_0)$ write $L_h := \{1/2 - h, 1/2 + h\}$. Suppose $\mathcal{A}$ to be a (probabilistic) testing algorithm, which, provided the fact $\boldsymbol{p} \in \prod_{j\in J} L_{h_j}$ is known for some $\{h_j\}_{j\in J} \subsetneq (0, h_0)$ whereas the concrete value of $\boldsymbol{p}$ is unknown beforehand, is able to test $\mathbf{H}_{0;J}$ versus $\mathbf{H}_{1;J}$ with error probability at most γ. In other words, with $\mathbf{p}' := (1/2 + h_j)_{j\in J}$ we have*

$$\mathbb{P}_{\boldsymbol{p}'}(\mathbf{D}(\mathcal{A}) = 0) \geq 1 - \gamma \quad \textit{and} \quad \forall \boldsymbol{p} \in \prod_{j\in J} L_{h_j} \setminus \{\boldsymbol{p}'\} : \mathbb{P}_{\boldsymbol{p}}(\mathbf{D}(\mathcal{A}) = 1) \geq 1 - \gamma.$$

Then, there exist some constant $c = c(h_0, \gamma_0) > 0$, which does not depend on h,γ or m, such that the corresponding stopping time $T^{\mathcal{A}}$ of $\mathcal{A}$ fulfills

$$\mathbb{E}_{\mathbf{p}'}[T^{\mathcal{A}}] \geq c \sum_{j\in J} h_j^{-2} \ln(\gamma^{-1}).$$

[6]Here, the expectation $\mathbb{E}_{\boldsymbol{p}}[T^{\mathcal{A}}]$ is also taken with regard to the possibly probabilistic behavior of $\mathcal{A}$, i.e., formally we have $\mathbb{E}_{\boldsymbol{p},\mathcal{A}}[T^{\mathcal{A}}]$. Nevertheless, we may simply write $\mathbb{E}_{\boldsymbol{p}}[T^{\mathcal{A}}]$ for convenience.

Moreover, for arbitrary $j \in J$, we have with $\mathbf{p}^{(j)} = (p^{(j)}_{j'})_{j' \in J}$ defined via $p^{(j)}_j = 1/2 - h_j$ and $p^{(j)}_{j'} = 1/2 + h_{j'}$ for all $j' \neq j$ that

$$\mathbb{E}_{\mathbf{p}^{(j)}}[T^{\mathcal{A}}] \geq c h_j^{-2} \ln(\gamma^{-1}).$$

Proof. At first, note that the case $|J| = 1$ of Lemma G.1 corresponds to the testing problem considered in Lemma B.1. For the sake of convenience, suppose without loss of generality $J = \{1, \ldots, N\}$. For an algorithm $\mathcal{A}'$ with sample access to $(p_1, \ldots, p_N)$, write $T^{\mathcal{A}}_j$ for the number of times $\mathcal{A}'$ queries the coin C_j (with bias p_j) until termination. Moreover, for $j \in [N]$ and $p \in [0, 1]$ define $\mathbf{p}^{[j]}(p) = (p^{[j]}_{j'}(p))_{j' \in J}$ where $p^{[j]}_j(p) = p$ and $p^{[j]}_{j'} = 1/2 + h_{j'}$ for $j' \neq j$.
Now, suppose $\mathcal{A}$, $\mathbf{p}'$ and $\mathbf{p}^{(j)}$ to be as in the statement of this Lemma. Let $\mathcal{A}(j)$ be the algorithm, which is given sample access to p_j as input, simulates $\mathcal{A}$ with sample access to $\mathbf{p}^{[j]}(p_j)$ as input, terminates as soon as $\mathcal{A}$ terminates and outputs 0 if $\mathcal{A}$ outputs 0 and outputs 1 if $\mathcal{A}$ outputs 1. As $\mathcal{A}$ is able to decide $\mathbf{H}_{0;J} : \forall j \in J : p_j > 1/2$ versus $\mathbf{H}_{1;J} : \exists j \in J : p_j < 1/2$ with error probability $\leq \gamma$ for every $\mathbf{p} \in \prod_{j \in J} L_{h_j}$, $\mathcal{A}(j)$ is able to decide whether $p_j > 1/2$ or $p_j < 1/2$ with error probability $\leq \gamma$ in both cases $p_j = 1/2 + h_j$ and $p_j = 1/2 - h_j$. Lemma B.1 assures that $\mathcal{A}(j)$ fulfills

$$\mathbb{E}_{1/2+h_j}[T^{\mathcal{A}(j)}] \geq c h_j^{-2} \ln(\gamma^{-1}) \quad \text{and} \quad \mathbb{E}_{1/2-h_j}[T^{\mathcal{A}(j)}] \geq c h_j^{-2} \ln(\gamma^{-1})$$

with $c = c(h_0, \gamma_0) > 0$ as in Lemma B.1, where $T^{\mathcal{A}(j)}$ denotes the number of times algorithm $\mathcal{A}(j)$ with sample access to $\mathbf{p}^{[j]}(p_j)$ queries any of the coins $C_1, \ldots, C_N$ before termination. As deciding whether $p_j > 1/2$ or $p_j < 1/2$ does not require knowledge about any of the coins $C_{j'}$, $j' \neq j$, which are independent of C_j, we may assume without loss of generality that $\mathcal{A}(j)$ throws *only* coin C_j for this purpose.[7] Regarding that $\mathbf{p}^{[j]}(1/2 + h_j) = \mathbf{p}'$ and $\mathbf{p}^{[j]}(1/2 - h_j) = \mathbf{p}^{(j)}$ hold, we obtain

$$\mathbb{E}_{\mathbf{p}'}[T^{\mathcal{A}}_j] = \mathbb{E}_{1/2+h_j}[T^{\mathcal{A}(j)}] \geq c h_j^{-2} \ln(\gamma^{-1}) \quad \text{and} \quad \mathbb{E}_{\mathbf{p}^{(j)}}[T^{\mathcal{A}}] \geq \mathbb{E}_{\mathbf{p}^{(j)}}[T^{\mathcal{A}}_j] = \mathbb{E}_{1/2-h_j}[T^{\mathcal{A}(j)}] \geq c h_j^{-2} \ln(\gamma^{-1}).$$

As this holds for each $j \in [N]$ we get

$$\mathbb{E}_{\mathbf{p}'}[T^{\mathcal{A}}] = \sum\nolimits_{j \in [N]} \mathbb{E}_{\mathbf{p}'}[T^{\mathcal{A}}_j] \geq c \sum\nolimits_{j \in [N]} h_j^{-2} \ln(\gamma^{-1}),$$

which completes the proof.

□

We proceed with the proof of Theorem F.2.

Proof of Theorem F.2. (i) As any (probabilistic) algorithm $\mathcal{A}$, which solves **Verify_i_as_**CW$\{\exists\text{CW}\}$ on $\mathcal{Q}^h_m$ with the guarantees (7), trivially also fulfills the weaker guarantees

$$\inf\nolimits_{\mathbf{Q} \in \mathcal{Q}^h_m(\text{CW}): \text{CW}(\mathbf{Q}) = i} \mathbb{P}_{\mathbf{Q}}(\mathbf{D}(\mathcal{A}[i]) = i) \geq 1 - \max\{\alpha, \beta\} \quad \text{and}$$
$$\inf\nolimits_{\mathbf{Q} \in \mathcal{Q}^h_m : \mathbf{Q} \in \mathcal{Q}_m(\neg\text{CW}) \text{ or } \text{CW}(\mathbf{Q}) \neq i} \mathbb{P}_{\mathbf{Q}}(\mathbf{D}(\mathcal{A}[i]) = \neg i) \geq 1 - \max\{\alpha, \beta\}$$

for every $i \in [m]$, we may suppose w.l.o.g. $\alpha = \beta =: \gamma$ from now on.
Let $i \in [m]$ be fixed. Choose $J := \{i\} \times ([m] \setminus \{i\})$. As $\mathcal{A}$ solves **Verify_i_as_**CW$\{\exists\text{CW}\}$, $\mathcal{A}[i]$ is able to decide[8]

$$\mathbf{H}_{0;J} : \forall j \in [m] \setminus \{i\} : p_{i,j} \geq 1/2 \qquad \mathbf{H}_{1;J} : \exists j \in [m] \setminus \{i\} : p_{i,j} < 1/2$$

for each $\boldsymbol{p} = (p_{i,j})_{j \in [m] \setminus \{i\}} \in \prod_{j \in [m] \setminus \{i\}} \{1/2 \pm h_{i,j}\}$ with error probability $\leq \gamma$. If $\mathbf{Q} \in \mathcal{Q}^h_m(\text{CW})$ with $\text{CW}(\mathbf{Q}) = i$, then $q_{i,j} = 1/2 + h_{i,j} > 1/2$ for every $(i, j) \in J$ and thus Lemma G.1 implies

$$\mathbb{E}_{\mathbf{Q}}[T^{\mathcal{A}[i]}] \geq c \sum\nolimits_{j \in [m] \setminus \{i\}} h_{i,j}^{-2} \ln(\gamma^{-1})$$

[7] To see this formally, suppose on the contrary that $T^{\mathcal{A}(j)}_j < c h^{-2} \ln(\gamma^{-1})$. Let $\widetilde{\mathcal{A}}(j)$ be given sample access to $\mathbf{p}^{[j]}(p_j)$ and behave as $\mathcal{A}(j)$ with the only difference that samples from any coin $C_{j'} \neq C_j$ are replaced by an artificial sample $\text{Ber}(p_{j'})$, which is independent of all the coins. Then, none of the coins $C_{j'} \neq C_j$ are thrown, we have $T^{\widetilde{\mathcal{A}}(j)} = T^{\widetilde{\mathcal{A}}(j)}_j$ and thus $\mathbb{E}_{1/2+h_j}[T^{\widetilde{\mathcal{A}}(j)}] = \mathbb{E}_{1/2+h_j}[T^{\widetilde{\mathcal{A}}(j)}_j] = \mathbb{E}_{1/2+h_j}[T^{\mathcal{A}(j)}_j] < c h^{-2} \ln(\gamma^{-1})$, a contradiction to Lemma B.1.

[8] Note here that for every $i \in [m]$ and $(p_{i,j})_{j \in [m] \setminus \{i\}} \in \prod_{j \in [m] \setminus \{i\}} \{1/2 \pm h_{i,j}\}$ with $\min_{i \neq j} h_{i,j} > h$, there is a $\mathbf{Q}' = (q'_{i',j'})_{1 \leq i',j' \leq m} \in \mathcal{Q}^h_m(i)$ with $|q'_{i,j} - 1/2| = |p_{i,j} - 1/2|$ for every $j \neq i$.

with $c = c(h_0, \gamma_0)$ as in Lemma G.1. By choosing $\mathbf{Q}(\varepsilon) \in \mathcal{Q}_m^h(\mathrm{CW})$ with $\mathrm{CW}(\mathbf{Q}(\varepsilon)) = i$ such that $|q_{i',j'} - 1/2| = h + \varepsilon$ for all $(i', j') \in (m)_2$ and arbitrarily small $\varepsilon > 0$ we obtain

$$\sup_{\mathbf{Q}\in\mathcal{Q}_m^h} \mathbb{E}_{\mathbf{Q}}[T^{\mathcal{A}[i]}] \geq \mathbb{E}_{\mathbf{Q}(\varepsilon)}[T^{\mathcal{A}[i]}] \geq \frac{c(m-1)\ln(\gamma^{-1})}{(h+\varepsilon)^2},$$

whence taking the limit $\varepsilon \searrow 0$ completes the proof of (i).

(ii) As in part (i), we may suppose w.l.o.g. $\alpha = \beta =: \gamma$ from now on. As $\mathbf{Q} = (q_{i,j})_{1\leq i,j\leq m}$ has a CW iff $(q_{\sigma(i),\sigma(j)})_{1\leq i,j\leq m} \in \mathcal{Q}_m$ has a CW, we may suppose w.l.o.g. $\sigma = \mathrm{id}$ in the following, i.e., $\mathrm{CW}(\mathbf{Q}) = 1$ and $q_{i,i+1} > 1/2$ for every $i \in [m-1]$. For $\mathbf{p} = (p_{1,3}, \ldots, p_{1,m}) \in [0,1]^{m-2}$ define $\hat{\mathbf{Q}}(\mathbf{p}) \in \mathcal{Q}_m$ via

$$\hat{\mathbf{Q}}(\mathbf{p})_{i,j} = \begin{cases} p_{i,j}, & \text{if } i = 1 \text{ and } j \in \{3, \ldots, m\}, \\ q_{i,j}, & \text{otherwise}, \end{cases}$$

for any $1 \leq i < j \leq m$. As $\min_{i\in[m-1]} q_{i,i+1} > 1/2$ by assumption on σ, for any $\mathbf{p} \in [0,1]^{m-2}$ either $\hat{\mathbf{Q}}(\mathbf{p}) \in \mathcal{Q}_m(\neg\mathrm{CW})$ or $\mathrm{CW}(\hat{\mathbf{Q}}(\mathbf{p})) = 1$ is fulfilled. Provided $\mathbf{p} \in ([0,1/2) \cup (1/2,1])^{m-2}$, we thus have the equivalence

$$\hat{\mathbf{Q}}(\mathbf{p}) \in \mathcal{Q}_m(\mathrm{CW}) \;\Leftrightarrow\; \forall j \in \{3, \ldots, m\} : p_{1,j} > 1/2.$$

Suppose $\mathcal{A}'$ to be the algorithm, which gets as input sample access to $\mathbf{p} \in [0,1]^{m-2}$, simulates $\mathcal{A}$ on $\hat{\mathbf{Q}}(\mathbf{p})$ and then outputs 0 if $\mathbf{D}(\mathcal{A}) = \mathrm{CW}$, and 1 if $\mathbf{D}(\mathcal{A}) = \neg\mathrm{CW}$. As $\mathcal{A}$ solves **Check_**CW on $\mathcal{Q}_m^h$ for γ, the algorithm $\mathcal{A}'$ is able to decide

$$\mathbf{H}_0 : \forall j \in \{3, \ldots, m\} : p_{1,j} \geq 1/2 \quad \text{versus} \quad \mathbf{H}_1 : \exists j \in \{3, \ldots, m\} : p_{1,j} < 1/2$$

with error probability at most γ for every $\mathbf{p} \in ([0, 1/2-h) \cup (1/2+h, 1])^{m-2}$. Regarding that $\mathbf{Q} \in \mathcal{Q}_m^h(\mathrm{CW})$ with $\mathrm{CW}(\mathbf{Q}) = 1$ implies $\mathbf{p}' := (q_{1,3}, \ldots, q_{1,m}) = (1/2 + h_{1,3}, \ldots, 1/2 + h_{1,m}) \in (1/2+h, 1]^{m-2}$, Lemma G.1 yields

$$\mathbb{E}_{\mathbf{Q}}[T^{\mathcal{A}}] = \mathbb{E}_{\mathbf{p}'}\left[T^{\mathcal{A}'}\right] \geq c\sum\nolimits_{j=3}^{m} h_{1,j}^{-2}\ln(\gamma^{-1}) \geq c\min\nolimits_{j\in[m]\setminus\{1\}} \sum\nolimits_{j'\in[m]\setminus\{1,j\}} h_{1,j'}^{-2}\ln(\gamma^{-1}).$$

The rest follows as in Part (i), i.e., by considering relations $\mathbf{Q}(\varepsilon)$ with entries in $\{1/2 \pm (h+\varepsilon)\}$ and taking the limit $\varepsilon \searrow 0$.

□

With Theorem F.2, the proof of Theorem 4.1 becomes trivial.

Proof of Theorem 4.1. Suppose $\mathbf{Q} \in \mathcal{Q}_m^h(\mathrm{CW})$ to be fixed and let $i^* := \mathrm{CW}(\mathbf{Q})$. If $\mathcal{A}$ solves the testification task for the CW on $\mathcal{Q}_m^h$ for α, β, then the algorithm $\widehat{\mathcal{A}}$ which takes any $i \in [m]$ as input, simulates $\mathcal{A}$ until it terminates and then outputs

$$\mathbf{D}(\widehat{\mathcal{A}}[i]) := \begin{cases} i, & \text{if } \mathbf{D}(\mathcal{A}) = \mathrm{CW}, \\ \neg i, & \text{if } \mathbf{D}(\mathcal{A}) = \neg\mathrm{CW} \text{ or } \mathbf{D}(\mathcal{A}) \in [m] \setminus \{i\}, \end{cases}$$

solves **Verify_i_as_**CW on $\mathcal{Q}_m^h$ for α and β. Therefore, the statement follows directly from Part (i) of Theorem F.2 with the choice $i = i^*$. □

H PROOF OF THEOREM 5.3 AND AN INSTANCE-WISE VERSION OF THEOREM 5.3

Proof of Theorem 5.3. Write $\mathcal{A} := \mathcal{A}^{\mathrm{NTS}}$ for the algorithm as considered in the statement of this Theorem. Note that the choices of the queries in Algorithm 6 can indeed be described by an appropriate sampling strategy π, i.e., $\mathcal{A}$ is of the form as stipulated by Algorithm 1 with $\alpha = \beta = \gamma$. Lines 19–21 in Algorithm 6 and the same argumentation as in the proof of Theorem 5.2 assure that $\mathcal{A}$ – and whence also π – fulfills $(\boldsymbol{n}_t)_{i,j} \to \infty$ almost surely for every $(i,j) \in (m)_2$. Thus, according to Theorem 5.2, $\mathcal{A}$ terminates a.s. (cf. the discussion in Section 5.4). Moreover, Lemma B.1 ensures that each duel proposed by $\mathcal{A}_{\mathrm{Bin}}$ is conducted in expectation at most $\mathcal{O}(h^{-2}\ln(\gamma^{-1}))$ times. Now, suppose $\mathcal{A}_{\mathrm{Bin}}$ to be testification-correct, i.e., an element of $\mathfrak{A}_m^*$. If $\mathcal{A}_{\mathrm{Bin}}$ terminates, then (due to its testification-correctness) we have according to Lemma E.1 that $\hat{G}_t \in \mathcal{G}_m(\neg\mathrm{CW}) \cup \bigcup_{i\in[m]} \mathcal{G}_m(i)$. Consequently, $\mathcal{A}$ terminates before reaching Line 19; at termination it has queried only those edges, which have been proposed by $\mathcal{A}_{\mathrm{Bin}}$, i.e., at most $T^{\mathcal{A}_{\mathrm{Bin}}}$ many. From this, we can directly infer that $\sup_{\mathbf{Q}\in\mathcal{Q}_m^h}[T^{\mathcal{A}}] \in \mathcal{O}(T^{\mathcal{A}_{\mathrm{Bin}}} h^{-2}\ln(\gamma^{-1}))$ as $\max\{h^{-1}, \gamma^{-1}\} \to \infty$. □

Algorithm 6 $\mathcal{A}^{\mathrm{NTS}}$ using $\mathcal{A}_{\mathrm{Bin}}$ as DSTA

Parameters: $\mathcal{A}_{\mathrm{Bin}} \in \mathfrak{A}_m$, $h \in (0, 1/2)$, $\gamma \in (0, 1)$

Initialization: Let $\hat{E}_0 \leftarrow \emptyset$, $\boldsymbol{n}_0 \leftarrow (0)_{1 \le i,j \le m}$, $\boldsymbol{w}_0 \leftarrow (0)_{1 \le i,j \le m}$, $t' \leftarrow 1$, $t \leftarrow 1$ and $\gamma' \leftarrow \frac{\gamma}{m}$, $C_{h,\gamma'}$ as in (7)

▷ $\hat{E}_t$ = set of edges (i, j), for which $q_{i,j} > \frac{1}{2}$ w.h.p.
▷ t = number of observations for π
▷ t' = number of observations for $\mathcal{A}_{\mathrm{Bin}}$
▷ $1 - \gamma'$ = confidence to which each pair is tested

1: $(i, j) \leftarrow (i^{\mathcal{A}_{\mathrm{Bin}}}(1), j^{\mathcal{A}_{\mathrm{Bin}}}(1))$ ▷ Get first query of $\mathcal{A}_{\mathrm{Bin}}$
2: **while** $\mathcal{A}_{\mathrm{Bin}}$ did not terminate yet **do**
3: Observe $X_{i,j}^{[t]} \sim \mathrm{Ber}(q_{i,j})$
4: Define $\boldsymbol{w}_t$ via $(\boldsymbol{w}_t)_{k,l} \leftarrow (\boldsymbol{w}_{t-1})_{k,l} + \mathbf{1}_{\{\{k,l\}=\{i,j\} \text{ and } X_{k,l}^{[t]}=1\}}$ $\forall 1 \le k, l \le m$
5: Define $\boldsymbol{n}_t$ via $(\boldsymbol{n}_t)_{k,l} \leftarrow (\boldsymbol{n}_{t-1})_{k,l} + \mathbf{1}_{\{\{k,l\}=\{i,j\}\}}$ $\forall 1 \le k, l \le m$
6: $\hat{E}_t \leftarrow \hat{E}_{t-1}$
7: **if** $(\hat{\boldsymbol{q}}_t)_{i,j} > 1/2 + C_{h,\gamma'}((\boldsymbol{n}_t)_{i,j})$ **then** ▷ $i \to j$ in $G_{\mathbf{Q}}$ w.h.p.
8: $\hat{E}_t \leftarrow \hat{E}_t \cup \{(i, j)\}$
9: Forward 1 to $\mathcal{A}_{\mathrm{Bin}}$ and set $t' \leftarrow t' + 1$ ▷ $\mathcal{A}_{\mathrm{Bin}}$ observes $i^{\mathcal{A}_{\mathrm{Bin}}}(t') \to j^{\mathcal{A}_{\mathrm{Bin}}}(t')$
10: Let $(i, j) \leftarrow (i^{\mathcal{A}_{\mathrm{Bin}}}(t'), j^{\mathcal{A}_{\mathrm{Bin}}}(t'))$ ▷ Choose next query from $\mathcal{A}_{\mathrm{Bin}}$
11: **else if** $(\hat{\boldsymbol{q}}_t)_{i,j} < 1/2 - C_{h,\gamma'}((\boldsymbol{n}_t)_{i,j})$ **then** ▷ $j \to i$ in $G_{\mathbf{Q}}$ w.h.p.
12: $\hat{E}_t \leftarrow \hat{E}_t \cup \{(j, i)\}$
13: Forward 0 to $\mathcal{A}_{\mathrm{Bin}}$ and set $t' \leftarrow t' + 1$ ▷ $\mathcal{A}_{\mathrm{Bin}}$ observes $j^{\mathcal{A}_{\mathrm{Bin}}}(t') \to i^{\mathcal{A}_{\mathrm{Bin}}}(t')$
14: Let $(i, j) \leftarrow (i^{\mathcal{A}_{\mathrm{Bin}}}(t'), j^{\mathcal{A}_{\mathrm{Bin}}}(t'))$ ▷ Choose next query from $\mathcal{A}_{\mathrm{Bin}}$
15: $t \leftarrow t + 1$
16: **if** $\exists i^* \in [m] : \hat{G}_t \in \mathcal{G}_m(i^*)$ **then**
17: **return** i^*
18: **if** $\hat{G}_t \in \mathcal{G}_m(\neg\mathrm{CW})$ **then return** $\neg\mathrm{CW}$
19: **while** True **do** ▷ No interaction with $\mathcal{A}_{\mathrm{Bin}}$ anymore
20: Sample a pair (i, j) uniformly at random from $\langle m \rangle_2$.
21: Do Steps 3–8, 11, 12 and 15–18

Without much effort, we obtain the following instance-wise bound for the expected runtime of Algorithm 6. Similarly, one may obtain an instance-wise expected runtime bound of Algorithm 5, which solves **Verify_i_as_**CW.

Theorem H.1. *Let $\alpha, \beta \in (0, 1/2)$ and $h \in (0, 1/2)$ be fixed. Suppose $\mathcal{A}_{\mathrm{Bin}} \in \mathfrak{A}_m^*$ and let $\mathcal{A} = \mathcal{A}^{\mathrm{NTS}}$ be Algorithm 6 called with the parameters h, α, β and $\mathcal{A}_{\mathrm{Bin}}$ as its black-box DSTA. Then, $\mathcal{A}$ solves the testification problem for the CW on $\mathcal{Q}_m^h$ for α and β. Define $\gamma' = \min\{\frac{\alpha}{m}, \frac{\beta}{m-1}\}$, fix an arbitrary $\mathbf{Q} \in \mathcal{Q}_m^h$ and let $h_{i,j} = |1/2 - q_{i,j}|$ for each distinct $i, j \in [m]$. Suppose $(i_1, j_1), \ldots, (i_{\binom{m}{2}}, j_{\binom{m}{2}}) \in (m)_2$ to be distinct and such that $h_{i_1,j_1} \le \cdots \le h_{i_{\binom{m}{2}}, j_{\binom{m}{2}}}$ holds. Then, we have with $c(h, \gamma') := \left\lceil \frac{\ln((1-\gamma')/\gamma')}{\ln((1/2+h)/(1/2-h))} \right\rceil$ that*

$$\mathbb{E}_{\mathbf{Q}}[T^{\mathcal{A}}] \le c(h, \gamma') \sum_{k=1}^{T^{\mathcal{A}_{\mathrm{Bin}}}} (2h_{i_k,j_k})^{-1} \left| 1 - 2\left(1 + (1/2 + h_{i_k,j_k})^{c(h,\gamma')} (1/2 - h_{i_k,j_k})^{-c(h,\gamma')}\right)^{-1} \right|.$$

Proof. Similarly as in the proof of Theorem 5.3 we see that $\mathcal{A}$ solves the testification problem for the CW on $\mathcal{Q}_m^h$ and also that it only queries those edges, which have been proposed by $\mathcal{A}_{\mathrm{Bin}}$. According to the choice of $C_{h,\gamma'}$ and the identity (3) stated in the proof of Lemma B.1, any such edge (i', j') proposed by $\mathcal{A}_{\mathrm{Bin}}$ is queried in expectation at most

$$\frac{c(h, \gamma')}{2h_{i',j'}} \left| 1 - 2\left(1 + (1/2 + h_{i',j'})^{c(h,\gamma')} (1/2 - h_{i',j'})^{-c(h,\gamma')}\right)^{-1} \right|$$

times by $\mathcal{A}$. This immediately concludes the proof. □

I FURTHER EXPERIMENTS

In this section, we provide further experiments for the algorithmic solutions developed in this work. In particular, Section I.1 resp. Section I.2 extend the experiments from Section 7 with regard to two important aspects, namely by considering preference relations with a larger number of arms resp. with or without a Condorcet winner. Finally, we provide in Section I.3 the Hudry tournament, which has been considered in the experiment of Section 7.2.

Throughout our experiments (this includes the experiments in Section 7 as well), we denote by $\mathcal{A}^{\mathrm{NTS}}$ (or simply NTS) Algorithm 6 initiated with Algorithm 3 (i.e., according to Proposition E.2 an optimal testification-correct DSTA) as $\mathcal{A}_{\mathrm{Bin}}$, and parameters m, h and $\alpha = \beta = \gamma$. The experiments in Sections 7, I.1 and I.2, which involved a variation of γ, were conducted with the values 0.001, 0.005, 0.01, 0.015, 0.02, 0.03, 0.05, 0.075, 0.1, 0.125, 0.15, 0.2, 0.25, 0.35, 0.45, 0.5, 0.55, 0.6, 0.65, 0.7, 0.75, 0.95 and 0.99 for γ.

I.1 LARGER NUMBER OF ARMS

Here, we repeat the experiment from Section 7.1 with $m = 8$ and $m = 10$ arms. That is, we sample uniformly at random relations $\mathbf{Q}$ from $\mathcal{Q}_m^{0.05}$, execute $\mathcal{A}^{\mathrm{NTS}}$ and SELECT-THEN-VERIFY with the same parameter h for different values of γ and plot the average termination time as well as the observed accuracy over 100000 repetitions of the experiments.

Figure 1 illustrates the effect of increasing the number of arms on the success rate and termination time of both algorithms. It is clearly visible that the larger the value of m, the larger both the success rate and the termination times of both algorithms. Again, $\mathcal{A}^{\mathrm{NTS}}$ apparently outperforms SELECT-THEN-VERIFY in terms of accuracy and sample complexity.

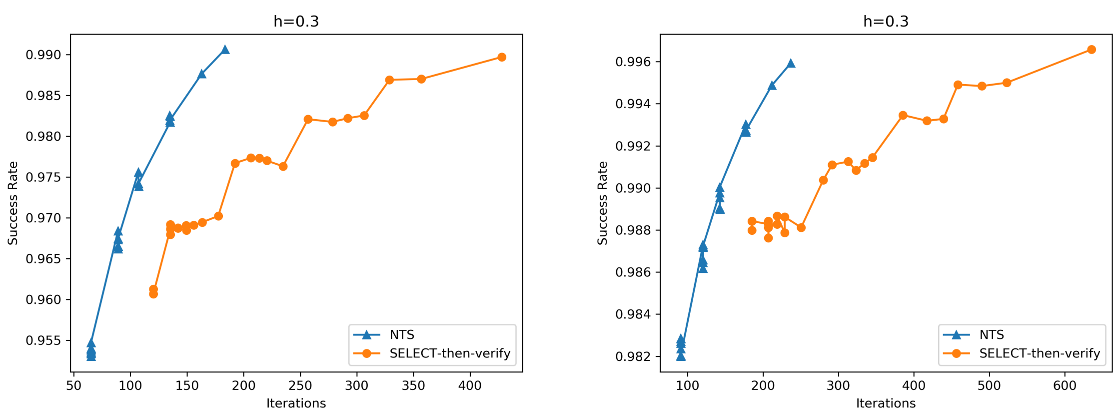


Figure 1: Accuracy and termination time of $\mathcal{A}^{\mathrm{NTS}}$ and SELECT-THEN-VERIFY for 8 arms (on the left) and 10 arms (on the right)

To further compare $\mathcal{A}^{\mathrm{NTS}}$ (from Cor. 5.4) with SELECT-THEN-VERIFY (StV), we conduct the following experiment: We fix $m \in \mathbb{N}$, $\gamma \in (0, 1/2)$ and $h \in (0, 1/2)$ in advance, sample relations $\mathbf{Q}_1, \ldots, \mathbf{Q}_N$ uniformly at random from $\widehat{\mathcal{Q}}_m^{h'} := \{\mathbf{Q} \in \mathcal{Q}_m \,|\, q_{i,j} \in \{1/2 \pm h'\} \forall (i,j) \in (m)_2\}$ and execute $\mathcal{A}^{\mathrm{NTS}}$ and SELECT-THEN-VERIFY with parameters m, h, γ on every instance $\mathbf{Q}_i$, $i \in \{1, \ldots, N\}$. Table 1 shows the observed mean sample complexities (with standard errors in brackets) as well as the accuracies of both algorithms for $N = 100$, $m = 20$, $\gamma = 0.05$ and $h = 0.05$ for different values of h'. Both algorithms achieve an accuracy of 100% for any $h' \geq h = 0.05$ and even for $h' = 0.02$. Moreover, $\mathcal{A}^{\mathrm{NTS}}$ clearly outperforms SELECT-THEN-VERIFY for any $h' > h$, and the magnitude to which extend this happens (i.e., the sample complexity gap) appears to be increasing in h'.

I.2 EXISTENCE OR NON-EXISTENCE OF A CONDORCET WINNER

Next, we repeat our experiment from Section 7.1 with the only difference that we sample $\mathbf{Q}$ uniformly at random from $\mathbf{Q}_5^{0.05}(\mathrm{CW})$ or from $\mathbf{Q}_5^{0.05}(\neg\mathrm{CW})$, respectively. As in the main part, the plots are generated by averaging over 25000 repetitions each. The results are shown in Figures 2 and 3 and demonstrate that $\mathcal{A}^{\mathrm{NTS}}$ outperforms SELECT-THEN-VERIFY in both cases.

Table 1: Experimental results for $m = 20, \gamma = 0.05, h = 0.05$, $N = 100$ and varying h'

	$T^{\mathcal{A}}$		Accuracy	
h'	$\mathcal{A}^{\mathrm{NTS}}$	StV	$\mathcal{A}^{\mathrm{NTS}}$	StV
0.45	**887** (4.1)	21751 (42.4)	1.00	1.00
0.40	**980** (5.4)	21730 (43.7)	1.00	1.00
0.35	**1085** (6.5)	21707 (48.1)	1.00	1.00
0.30	**1262** (8.5)	21773 (47.0)	1.00	1.00
0.25	**1447** (10.8)	21730 (52.3)	1.00	1.00
0.20	**1798** (15.6)	21832 (47.7)	1.00	1.00
0.15	**2331** (21.8)	21870 (51.0)	1.00	1.00
0.10	**3383** (29.3)	22096 (50.8)	1.00	1.00
0.05	**6607** (88.6)	22544 (92.3)	1.00	1.00
0.02	**14155** (234.6)	23567 (167.9)	1.00	1.00

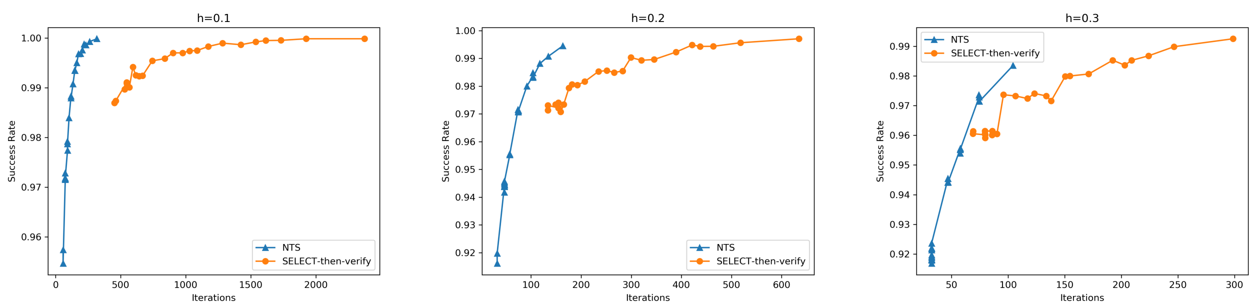


Figure 2: Accuracy and termination time of $\mathcal{A}^{\mathrm{NTS}}$ and SELECT-THEN-VERIFY for 5 arms provided a CW exists

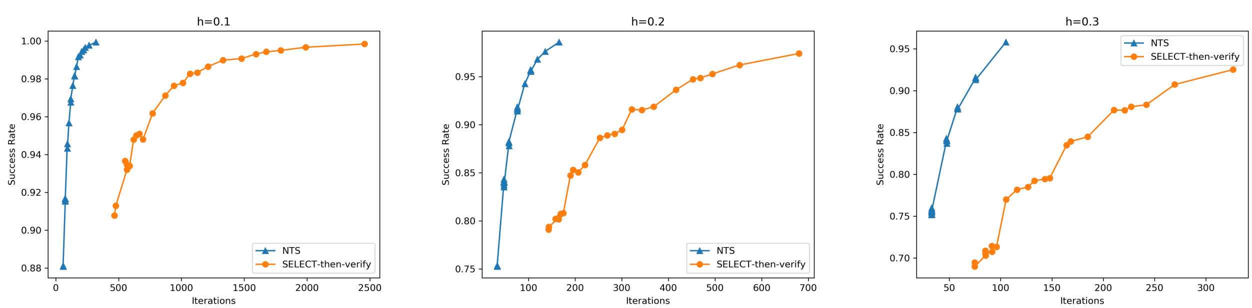


Figure 3: Accuracy and termination time of $\mathcal{A}^{\mathrm{NTS}}$ and SELECT-THEN-VERIFY for 5 arms provided a CW does not exist

I.3 DATA FOR THE PASSIVE SETTING

The preference relation $\mathbf{Q}_{\mathrm{Hudry}}$ corresponding to the Hudry-tournament (cf. [Ramamohan et al., 2016]), which is used in Section 7.2 is formally given as

$$\mathbf{Q}_{\text{Hudry}} := \begin{pmatrix} 0.5 & 0.1 & 0.1 & 0.1 & 0.6 & 0.6 & 0.6 & 0.6 & 0.6 & 0.6 & 0.6 & 0.6 & 0.6 \\ 0.9 & 0.5 & 0.9 & 0.1 & 0.1 & 0.1 & 0.1 & 0.9 & 0.9 & 0.9 & 0.9 & 0.9 & 0.9 \\ 0.9 & 0.1 & 0.5 & 0.9 & 0.9 & 0.9 & 0.9 & 0.1 & 0.1 & 0.1 & 0.9 & 0.9 & 0.9 \\ 0.9 & 0.9 & 0.1 & 0.5 & 0.9 & 0.9 & 0.9 & 0.9 & 0.9 & 0.9 & 0.1 & 0.1 & 0.1 \\ 0.4 & 0.9 & 0.1 & 0.1 & 0.5 & 0.9 & 0.9 & 0.9 & 0.9 & 0.9 & 0.1 & 0.1 & 0.1 \\ 0.4 & 0.9 & 0.1 & 0.1 & 0.1 & 0.5 & 0.9 & 0.9 & 0.9 & 0.9 & 0.1 & 0.1 & 0.1 \\ 0.4 & 0.9 & 0.1 & 0.1 & 0.1 & 0.1 & 0.5 & 0.9 & 0.9 & 0.9 & 0.1 & 0.1 & 0.1 \\ 0.4 & 0.1 & 0.9 & 0.1 & 0.1 & 0.1 & 0.1 & 0.5 & 0.9 & 0.9 & 0.9 & 0.9 & 0.9 \\ 0.4 & 0.1 & 0.9 & 0.1 & 0.1 & 0.1 & 0.1 & 0.1 & 0.5 & 0.9 & 0.9 & 0.9 & 0.9 \\ 0.4 & 0.1 & 0.9 & 0.1 & 0.1 & 0.1 & 0.1 & 0.1 & 0.1 & 0.5 & 0.9 & 0.9 & 0.9 \\ 0.4 & 0.1 & 0.1 & 0.9 & 0.9 & 0.9 & 0.9 & 0.1 & 0.1 & 0.1 & 0.5 & 0.9 & 0.9 \\ 0.4 & 0.1 & 0.1 & 0.9 & 0.9 & 0.9 & 0.9 & 0.1 & 0.1 & 0.1 & 0.1 & 0.5 & 0.9 \\ 0.4 & 0.1 & 0.1 & 0.9 & 0.9 & 0.9 & 0.9 & 0.1 & 0.1 & 0.1 & 0.1 & 0.1 & 0.5 \end{pmatrix}$$

J STICKY TRACK-AND-STOP FOR CHECK_CW AND TESTIFICATION FOR THE CW

Degenne and Koolen [2019] have proposed the STICKY TRACK-AND-STOP algorithm for the setting of pure exploration bandits with multiple correct answers, which also covers our problems of interest. In this section we explicitly state STICKY TRACK-AND-STOP for the problems of Testification for the CW and **Check_**CW and state and discuss their guarantees. We omit the problem **Verify_i_as_**CW here, because minor changes of the version for Testification for the CW also yields a solution to **Verify_i_as_**CW with similar guarantees. For the sake of convenience, we start with the easier problem **Check_**CW.

Let us define

$$\Delta_{(m)_2} := \left\{ (v_{i,j})_{1\le i<j\le m} \in \mathbb{R}^{\binom{m}{2}} \ : \ \sum\nolimits_{(i,j)\in(m)_2} v_{i,j} = 1 \text{ and } v_{i,j} \ge 0 \text{ for all } (i,j)\in(m)_2 \right\}$$

and for any $\varepsilon > 0$ also

$$\Delta^{\varepsilon}_{(m)_2} := \left\{ (v_{i,j})_{1\le i<j\le m} \in \Delta_{(m)_2} \ : \ v_{i,j} \ge \varepsilon \text{ for all } (i,j)\in(m)_2 \right\}.$$

In the following, let $d_{\text{KL}}(p,q) = p\ln(p/q) + (1-p)\ln((1-p)/(1-q))$ be the KL-divergence between two random variables $X \sim \text{Ber}(p)$ and $Y \sim \text{Ber}(q)$. For $\mathbf{v} = (v_{i,j})_{1\le i<j\le m} \in \Delta_{(m)_2}$ and $\mathbf{Q}, \mathbf{Q}' \in \mathcal{Q}_m$ let

$$D(\mathbf{v}, \mathbf{Q}, \mathbf{Q}') := \sum\nolimits_{(i,j)\in(m)_2} v_{i,j} d_{\text{KL}}(q_{i,j}, q'_{i,j}),$$

and for $\mathcal{Q}'_m \subseteq \mathcal{Q}_m$ let further

$$D(\mathbf{v}, \mathbf{Q}, \mathcal{Q}'_m) := \inf\nolimits_{\mathbf{Q}'\in\mathcal{Q}'_m} D(\mathbf{v}, \mathbf{Q}, \mathbf{Q}').$$

J.1 STICKY TRACK-AND-STOP FOR CHECK_CW

In the setting of **Check_**CW, the STICKY TRACK-AND-STOP algorithm from Degenne and Koolen [2019] can be stated as Algorithm 7.

Note that steps 2 and 11 of Algorithm 7 are already computationally expensive, but the calculation of $D(\mathbf{v}, \mathbf{Q}, \mathcal{Q}^h_m(\neg \text{X}))$ in step 5 is even more involved, especially in the case $\text{X} = \neg\text{CW}$, because $\mathcal{Q}^h_m(\text{CW})$ is non-convex[9]. Hence, the algorithm appears to be infeasible for practical applications to us.

For fixed $m \in \mathbb{N}$ and $h \in (0, 1/2)$ define for any $\mathbf{Q} \in \mathcal{Q}_m$ the value

$$\tilde{D}^h_m(\mathbf{Q}) := \begin{cases} \sup_{\mathbf{v}\in\Delta_{(m)_2}} D(\mathbf{v}, \mathbf{Q}, \mathcal{Q}^h_m(\neg\text{CW})), & \text{if } \mathbf{Q} \in \mathcal{Q}_m(\text{CW}), \\ \sup_{\mathbf{v}\in\Delta_{(m)_2}} D(\mathbf{v}, \mathbf{Q}, \mathcal{Q}^h_m(\text{CW})), & \text{if } \mathbf{Q} \in \mathcal{Q}_m(\neg\text{CW}). \end{cases}$$

This characteristic plays a crucial role in the theoretical results proven by Degenne and Koolen [2019], which we will state and comment on below in Proposition J.4. As a first step, we prove upper and lower bounds for $\tilde{D}^h_m(\mathbf{Q})$. For this purpose, we will make use of the following Lemma. It is taken from [Bubeck and Cesa-Bianchi, 2012] and is a mere consequence of Pinsker's theorem as well as the inequality $\ln x \le x - 1$, which holds for all $x > 0$.

[9] It is the union of disjoint convex sets.

Algorithm 7 : STICKY TRACK-AND-STOP FOR **CHECK**_CW

Input: $\gamma \in (0,1)$, $h \in (0,1/2)$, a sequence $(\varepsilon_t)_{t\in\mathbb{N}}$, functions $t \mapsto f(t)$ and $(t,\gamma) \mapsto \beta(t,\gamma)$
Initialization: $t \leftarrow 1$, $\hat{\boldsymbol{Q}}_0 \leftarrow (0)_{1\le i<j\le m}$, $\boldsymbol{n}_0 \leftarrow (0)_{1\le i,j\le m}$.

while True **do**
  Let $\mathcal{C}_t \leftarrow \{\mathbf{Q}' \in \mathcal{Q}_m^h : D(\boldsymbol{n}_{t-1}/(t-1), \hat{\mathbf{Q}}_{t-1}, \mathbf{Q}') \le \ln(f(t-1))\}$
  Compute $I_t = \{\mathrm{X} \in \{\mathrm{CW}, \neg\mathrm{CW}\} \mid \exists \mathbf{Q}' \in \mathcal{Q}_m^h(\mathrm{X}) \cap \mathcal{C}_t\}$
  Choose an element X from I_t, prefer CW over $\neg$CW
  Compute that weight $\mathbf{v}_t \in \Delta_{\langle m\rangle_2}$, which maximizes $D(\mathbf{v}_t, \hat{\mathbf{Q}}_{t-1}, \mathcal{Q}_m^h(\neg\mathrm{X}))$
  Compute the projection $\mathbf{v}_t^{\varepsilon_t}$ of $\mathbf{v}_t$ onto $\Delta_{\langle m\rangle_2}^{\varepsilon_t}$
  Pull $(i,j) = \mathrm{argmin}_{(i',j')\in\langle m\rangle_2} (\boldsymbol{n}_t)_{i',j'} - \sum_{s=1}^{t} (\mathbf{v}_s^{\varepsilon_s})_{i',j'}$, observe $X_{i,j} \sim \mathrm{Ber}(q_{i,j})$
  Update $\boldsymbol{w}_t$ via $(\boldsymbol{w}_t)_{k,l} \leftarrow (\boldsymbol{w}_{t-1})_{k,l} + \mathbf{1}_{\{\{k,l\}=\{i,j\} \text{ and } X_{k,l}=1\}}$ $\quad\forall 1 \le k,l \le m$
  Update $\boldsymbol{n}_t$ via $(\boldsymbol{n}_t)_{k,l} \leftarrow (\boldsymbol{n}_{t-1})_{k,l} + \mathbf{1}_{\{\{k,l\}=\{i,j\}\}}$ $\quad\forall 1 \le k,l \le m$
  Update $\hat{\mathbf{Q}}_t \leftarrow \frac{\boldsymbol{w}_t}{\boldsymbol{n}_t}$.
  Let $\mathcal{D}_t \leftarrow \{\mathbf{Q}' \in \mathcal{Q}_m^h : D(\boldsymbol{n}_t/t, \hat{\mathbf{Q}}_t, \mathbf{Q}') \le \beta(t,\gamma)\}$
  if $\exists \mathrm{X} \in \{\mathrm{CW}, \neg\mathrm{CW}\}$ with $\mathcal{D}_t \cap \mathcal{Q}_m^h(\neg\mathrm{X}) = \emptyset$ **then**
    return X
  Update $t \leftarrow t+1$

Lemma J.1. *For $p, q \in [0,1]$ we have*

$$2(p-q)^2 \le d_{\mathrm{KL}}(p,q) \le \frac{(p-q)^2}{q(1-q)}.$$

Moreover, we will make use of the following result, which follows immediately by the definition of the Condorcet winner.

Lemma J.2. *Suppose $\mathbf{Q} \in \mathcal{Q}_m(\mathrm{CW})$ with $i = \mathrm{CW}(\mathbf{Q})$, $j \in [m] \setminus \{i\}$ and let $\mathbf{Q}' = (q'_{i,j})_{1\le i,j\le m}$ be defined via $q'_{i,j} = 1 - q_{i,j}$ and $q'_{i',j'} = q_{i',j'}$ for every $(i',j') \in \langle m\rangle_2 \setminus \{(i,j),(j,i)\}$. Then, either $j = \mathrm{CW}(\mathbf{Q}')$ or $\mathbf{Q}' \in \mathcal{Q}_m(\neg\mathrm{CW})$.*

We obtain the following lower and upper bounds for $\tilde{D}_m^h(\mathbf{Q})$. Note that the factor $m-2$ in (i) therein is in accordance with the factor $m-2$ in our lower bound for solutions to **Check**_CW from Theorem F.2.

Lemma J.3. *(i) For any $\mathbf{Q} \in \mathcal{Q}_m^h(\mathrm{CW})$ we have*

$$\tilde{D}_m^h(\mathbf{Q}) \le \frac{2d_h(\mathbf{Q})}{m-2}$$

with $d_h(\mathbf{Q}) \coloneqq \max_{(i,j)\in\langle m\rangle_2} \max\{d_{\mathrm{KL}}(q_{i,j}, 1/2+h), d_{\mathrm{KL}}(q_{i,j}, 1/2-h)\}$.

(ii) For any $\mathbf{Q} \in \mathcal{Q}_m^h$ we have

$$\tilde{D}_m^h(\mathbf{Q}) \ge \frac{8h^2}{m}. \tag{9}$$

Proof. (i) Let $\mathbf{Q} \in \mathcal{Q}_m^h(\mathrm{CW})$ be fixed, and suppose $\mathbf{v} \in \Delta_{\langle m\rangle_2}$ to be fixed for the moment. Let $i \coloneqq \mathrm{CW}(\mathbf{Q})$. By assumption on $\mathbf{v}$ there exists[10] some distinct $j', j'' \in [m] \setminus \{i\}$ with $\max\{v_{i,j'}, v_{i,j''}\} \le 2(m-2)^{-1}$. According to Lemma J.2 we can choose $j \in \{j', j''\}$ such that $q_{j,k} < 1/2$ for at least one $k \in [m] \setminus \{i\}$. Thus, for arbitrarily small $\delta \in (0, 1/2-h)$, $\mathbf{Q}' \in \mathcal{Q}_m^h$ defined via

$$q'_{r,s} \coloneqq \begin{cases} 1/2 - (h+\delta), & \text{if } (r,s) = (i,j), \\ 1/2 + h + \delta, & \text{if } (r,s) = (j,i), \\ q_{r,s}, & \text{otherwise,} \end{cases}$$

[10]Indeed, as $\sum_{j'\neq i} v_{i,j'} \le 1$ one can choose $j' \neq i$ with $v_{i,j'} \le (m-1)^{-1}$. Now, $\sum_{j''\notin\{i,j'\}} v_{i,j''} \le 1$ allows us to choose $j'' \in [m] \setminus \{i, j'\}$ with $v_{i,j''} \le (m-2)^{-1}$. Then, $\max\{v_{i,j'}, v_{i,j''}\} \le v_{i,j'} + v_{i,j''} \le 2(m-2)^{-1}$ holds.

for each $(r,s) \in (m)_2$, fulfills $\mathbf{Q}' \in \mathcal{Q}_m^h(\neg\mathrm{CW})$. As $q_{i,j} > 1/2 + h$ holds by assumption on $\mathbf{Q}$ and i, the definition of $\mathbf{Q}'$ assures

$$\sum\nolimits_{(r,s)\in(m)_2} v_{r,s} d_{\mathrm{KL}}(q_{r,s}, q'_{r,s}) \leq v_{i,j} d_{\mathrm{KL}}(q_{i,j}, q'_{i,j}) \leq \frac{2 d_{\mathrm{KL}}(q_{i,j}, 1/2 - (h+\delta))}{m-2}.$$

Regarding $\mathbf{Q}' \in \mathcal{Q}_m^h(\neg\mathrm{CW})$ and that this estimate is obtained for any $\mathbf{v} \in \Delta_{(m)_2}$, we can conclude that

$$\tilde{D}_m^h(\mathbf{Q}) = \sup\nolimits_{\mathbf{v}\in\Delta_{(m)_2}} \inf\nolimits_{\mathbf{Q}'\in\mathcal{Q}_m^h(\neg\mathrm{CW})} \sum\nolimits_{(r,s)\in(m)_2} v_{r,s} d_{\mathrm{KL}}(q_{r,s}, q'_{r,s}) \leq \frac{2 d_{\mathrm{KL}}(q_{i,j}, 1/2 - (h+\delta))}{m-2}.$$

Taking the limit $\delta \searrow 0$ yields

$$\tilde{D}_m^h(\mathbf{Q}) \leq \frac{2 d_{\mathrm{KL}}(q_{i,j}, 1/2 - h)}{m-2} \leq \frac{2 d_h(\mathbf{Q})}{m-2}.$$

(ii) Let $\mathbf{Q} \in \mathcal{Q}_m^h$ be fixed. We distinguish two cases.
Case 1: $\mathbf{Q} \in \mathcal{Q}_m^h(\mathrm{CW})$. Let $i \coloneqq \mathrm{CW}(\mathbf{Q})$ and define $\mathbf{v} = (v_{r,s})_{(r,s)\in(m)_2} \in \Delta_{(m)_2}$ via $v_{r,s} = \mathbf{1}_{\{i\in\{r,s\}\}}(m-1)^{-1}$ for each $(r,s) \in (m)_2$. For any $\mathbf{Q}' \in \mathcal{Q}_m^h(\neg\mathrm{CW})$ there exists some $j \in [m] \setminus \{i\}$ with $q'_{i,j} < 1/2 - h$, as otherwise $\mathrm{CW}(\mathbf{Q}') = i$ would hold. But by assumption on i, $q_{i,j} > 1/2 + h$ holds, whence we can estimate with Lemma J.1 that

$$\sum\nolimits_{(r,s)\in(m)_2} v_{r,s} d_{\mathrm{KL}}(q_{r,s}, q'_{r,s}) \geq v_{i,j} d_{\mathrm{KL}}(q_{i,j}, q'_{i,j}) \geq \frac{d_{\mathrm{KL}}(1/2+h, 1/2-h)}{m-1} \geq \frac{8h^2}{m-1}.$$

As this holds for arbitrary $\mathbf{Q}'$, (9) follows.
Case 2: $\mathbf{Q} \in \mathcal{Q}_m^h(\neg\mathrm{CW})$. For every $i \in [m]$, $\mathrm{CW}(\mathbf{Q}) \neq i$ implies the existence of some $j(i) \in [m] \setminus \{i\}$ with $q_{i,j(i)} < 1/2 - h$. Now, choose $\mathbf{v} = (v_{r,s})_{(r,s)\in(m)_2} \in \Delta_{(m)_2}$ such that

$$v_{r,s} = \begin{cases} \frac{1}{m}, & \text{if } (r,s) \in \{(i,j(i)),(j(i),i)\} \text{ for some } i \in [m], \\ 0, & \text{otherwise,} \end{cases}$$

for any $(r,s) \in (m)_2$. Let $\mathbf{Q}' \in \mathcal{Q}_m^h(\mathrm{CW})$ be arbitrary and write $i' = \mathrm{CW}(\mathbf{Q}')$. Then, $q'_{i',j(i')} > 1/2 + h$ but at the same time $q_{i',j(i')} < 1/2 - h$ holds. Therefore, assuming for convenience w.l.o.g.[11] $i' < j(i')$, we obtain again with the help of Lemma J.1 the estimate

$$\sum\nolimits_{(r,s)\in(m)_2} v_{r,s} d_{\mathrm{KL}}(q_{r,s}, q'_{r,s}) \geq v_{i',j(i')} d_{\mathrm{KL}}(q_{i',j(i')}, q'_{i',j(i')}) \geq \frac{d_{\mathrm{KL}}(1/2+h, 1/2-h)}{m} \geq \frac{8h^2}{m}.$$

As $\mathbf{Q}'$ was arbitrary, we obtain (9).

□

Equipped with these results, we show the following:

Proposition J.4. *(i) Let $h \in (0,1/2)$ and $m \in \mathbb{N}$ be fixed. Any algorithm $\mathcal{A}$, which is able to test for any parameter $\gamma > 0$ (write $\mathcal{A}(\gamma)$) and any $\mathbf{Q} \in \mathcal{Q}_m^h$ with error probability at most γ whether $\mathbf{Q}$ is in $\mathcal{Q}_m(\mathrm{CW})$ or not, fulfills*

$$\liminf_{\gamma\to 0} \frac{\mathbb{E}_{\mathbf{Q}}[T^{\mathcal{A}(\gamma)}]}{\ln(\gamma^{-1})} \geq \frac{1}{\tilde{D}_m^h(\mathbf{Q})}$$

and consequently

$$\sup_{\mathbf{Q}\in\mathcal{Q}_m^h(\mathrm{CW})} \liminf_{\gamma\to 0} \frac{\mathbb{E}_{\mathbf{Q}}\left[T^{\mathcal{A}(\gamma)}\right]}{\ln(\gamma^{-1})} \geq \frac{(m-2)(1/4-h^2)}{8h^2}.$$

(ii) Let $h \in (0,1/2)$ and $m \in \mathbb{N}$ be fixed. Choose $C > 0$ such that $C \geq e\sum_{t=1}^{\infty} t^{-2} \left(e/\binom{m}{2}\right)^{\binom{m}{2}} \left(\ln^2(Ct^2)\ln(t)\right)^{\binom{m}{2}}$ and let

$$\varepsilon_t \coloneqq \frac{1}{2}\left(\binom{m}{2}^2 + t\right)^{-\frac{1}{2}}, \quad f(t) \coloneqq Ct^{10} \quad \text{and} \quad \beta(t,\gamma) \coloneqq \ln(Ct^2\gamma^{-1})$$

[11] In case $i' > j(i')$ estimate the following sum by $v_{j(i'),i'} d_{\mathrm{KL}}(q_{j(i'),i'}, q'_{j(i'),i'})$ and argue analogously.

Write $\mathcal{A}(\gamma)$ for Algorithm 7 called with parameters $\gamma, h, (\varepsilon_t)_t, f$ and β. Then, $\mathcal{A}(\gamma)$ is able to test for any $\mathbf{Q} \in \mathcal{Q}_m^h$ with error probability at most γ whether $\mathbf{Q}$ is in $\mathcal{Q}_m(\mathrm{CW})$ or not, and fulfills

$$\lim_{\gamma \to 0} \frac{\mathbb{E}_{\mathbf{Q}}[T^{\mathcal{A}(\gamma)}]}{\ln(\gamma^{-1})} = \frac{1}{\tilde{D}_m^h(\mathbf{Q})}$$

for any $\mathbf{Q} \in \mathcal{Q}_m^h$. In particular,

$$\sup_{\mathbf{Q} \in \mathcal{Q}_m^h} \lim_{\gamma \to 0} \frac{\mathbb{E}_{\mathbf{Q}}\left[T^{\mathcal{A}(\gamma)}\right]}{\ln(\gamma^{-1})} \leq \frac{m}{8h^2}$$

Proof. (i) The first statement corresponds to Theorem 1 in [Degenne and Koolen, 2019]. Next, let $\mathbf{Q} = \mathbf{Q}(\delta) \in \mathcal{Q}_m^h(\mathrm{CW})$ be such that $q_{i,j} \in \{1/2 \pm (h+\delta)\}$ for all $(i,j) \in (m)_2$ and some arbitrarily small $\delta > 0$. It holds that

$$\begin{aligned} d_h(\mathbf{Q}) &= \max_{(i,j)\in(m)_2} \max\{d_{\mathrm{KL}}(q_{i,j}, 1/2+h), d_{\mathrm{KL}}(q_{i,j}, 1/2-h)\} \\ &= d_{\mathrm{KL}}(1/2+(h+\delta), 1/2-h) \leq \frac{4(h+\delta/2)^2}{1/4-h^2} \end{aligned}$$

where we have used Lemma J.1 in the last step. Thus, the second statement follows from part (i) of Lemma J.3 by taking the limit $\delta \searrow 0$.

(ii) Theorem 11 in [Degenne and Koolen, 2019] implies the first statement. For the choice of ε_t confer p. 7 in [Garivier and Kaufmann, 2016], for $f(t)$ see Lemma 14 on p. 9 in [Degenne and Koolen, 2019] and for $\beta(t,\gamma)$ see Theorem 10 on p. 6 in [Degenne and Koolen, 2019]. The second statement follows directly from the bound on $\tilde{D}_m^h(\mathbf{Q})$ stated in Lemma J.3.

□

J.2 STICKY TRACK-AND-STOP FOR CW TESTIFICATION

Recall that $\mathcal{Q}_m(k) = \{\mathbf{Q} \in \mathcal{Q}_m(\mathrm{CW}) \,|\, \mathrm{CW}(\mathbf{Q}) = k\}$ for any $k \in [m]$ and define $\mathcal{I} := \{\neg\mathrm{CW}, 1, \ldots, m\}$. Then, $\mathcal{Q}_m = \bigcup_{\mathrm{X}\in\mathcal{I}} \mathcal{Q}_m(\mathrm{X})$ is a disjoint union. For $\mathrm{X} \in \mathcal{I}$ write $\mathcal{Q}_m(\neg\mathrm{X}) := \bigcup_{\mathrm{X}'\in\mathcal{I}\setminus\{\mathrm{X}\}} \mathcal{Q}_m(\mathrm{X}')$ and note that this definition is consistent with $\mathcal{Q}_m(\neg(\neg\mathrm{CW})) = \mathcal{Q}_m(\mathrm{CW})$. Moreover, write as usual $\mathcal{Q}_m^h(\mathrm{X}) = \mathcal{Q}_m^h \cap \mathcal{Q}_m(\mathrm{X})$ and similarly $\mathcal{Q}_m^h(\neg\mathrm{X}) = \mathcal{Q}_m^h \cap \mathcal{Q}_m(\neg\mathrm{X}) = \mathcal{Q}_m^h \setminus \mathcal{Q}_m(\mathrm{X})$ for any $\mathrm{X} \in \mathcal{I}$. We endow $\mathcal{I}$ with the ordering $\succ_{\mathcal{I}}$ defined[12] via $1 \succ_{\mathcal{I}} 2 \succ_{\mathcal{I}} \cdots \succ_{\mathcal{I}} m \succ_{\mathcal{I}} \neg\mathrm{CW}$; This way, choosing, e.g., an element from $\{2, 3, \neg\mathrm{CW}\} \subset \mathcal{I}$ according to $\succ_{\mathcal{I}}$ means to choose 2. Let $\Delta_{(m)_2}$ and $\Delta^{\varepsilon}_{(m)_2}$ be defined as above. For $\mathbf{v} = (v_{i,j})_{1\leq i<j\leq m} \in \Delta_{(m)_2}$ and $\mathbf{Q}, \mathbf{Q}' \in \mathcal{Q}_m$ let again

$$D(\mathbf{v}, \mathbf{Q}, \mathbf{Q}') := \sum\nolimits_{(i,j)\in(m)_2} v_{i,j} d_{\mathrm{KL}}(q_{i,j}, q'_{i,j}),$$

and for $\mathcal{Q}'_m \subseteq \mathcal{Q}_m$ let further

$$D(\mathbf{v}, \mathbf{Q}, \mathcal{Q}'_m) := \inf\nolimits_{\mathbf{Q}'\in\mathcal{Q}'_m} D(\mathbf{v}, \mathbf{Q}, \mathbf{Q}').$$

For $\mathbf{Q} \in \mathcal{Q}_m$ define

$$i_F(\mathbf{Q}) := \mathrm{argmax}_{\mathrm{X}'\in\mathcal{I}} \left(\mathrm{X}' \mapsto \sup\nolimits_{\mathbf{v}\in\Delta_{(m)_2}} D(\mathbf{v}, \mathbf{Q}, \mathcal{Q}_m^h(\neg\mathrm{X}'))\right).$$

and note that $(\neg\mathrm{CW}) \notin i_F(\mathbf{Q})$ whenever $\mathbf{Q} \in \mathcal{Q}_m^h(\mathrm{CW})$ holds[13].

In the setting of testification for the CW, the STICKY TRACK-AND-STOP algorithm from Degenne and Koolen [2019] can be stated as Algorithm 8.

Steps 2 and 11 of Algorithm 8 are the same as in Algorithm 7 and thus similarly computationally very expensive, and analogously step 5 is expensive, in particular if $\mathrm{X} = \neg\mathrm{CW}$. Moreover, as there are $m+1$ possible answers for the testification problem whereas testing is a problem with a binary outcome, step 3 in Algorithm 8 is far more complex than the corresponding step in Algorithm 7. This step requires to calculate for each $\mathbf{Q} \in \mathcal{C}_t$ the set $i_F(\mathbf{Q}) \subseteq \mathcal{I}$, which is the set of maximizers of $\mathrm{X}' \mapsto \sup_{\mathbf{v}\in\Delta_{(m)_2}} D(\mathbf{v}, \mathbf{Q}, \mathcal{Q}_m^h(\neg\mathrm{X}'))$. Finding $i_F(\mathbf{Q})$ for one fixed $\mathbf{Q}$ already requires the solution

[12] Here, we merely have to choose *any* fixed ordering on $\mathcal{I}$, which one is not of importance.

[13] In fact, if $\mathbf{Q} \in \mathcal{Q}_m^h(\mathrm{CW})$ with $\mathrm{CW}(\mathbf{Q}) = k$, then $D(\mathbf{v}, \mathbf{Q}, \mathcal{Q}_m^h(\neg k)) > 0 = D(\mathbf{v}, \mathbf{Q}, \mathcal{Q}_m^h(\mathrm{CW}))$.

Algorithm 8 : STICKY TRACK-AND-STOP FOR TESTIFICATION FOR THE CW

Input: $\gamma \in (0,1)$, $h \in (0,1/2)$, a sequence $(\varepsilon_t)_{t\in\mathbb{N}}$, functions $t \mapsto f(t)$ and $(t,\gamma) \mapsto \beta(t,\gamma)$

Initialization: $t \leftarrow 1$, $\hat{\boldsymbol{Q}}_0 \leftarrow (0)_{1\le i<j\le m}$, $\boldsymbol{n}_0 \leftarrow (0)_{1\le i,j\le m}$.

while True **do**
  Let $\mathcal{C}_t \leftarrow \{\mathbf{Q}' \in \mathcal{Q}_m^h : D(\boldsymbol{n}_{t-1}/(t-1), \hat{\mathbf{Q}}_{t-1}, \mathbf{Q}') \le \ln(f(t-1))\}$
  Let $I_t = \bigcup_{\mathbf{Q}'\in\mathcal{C}_t} i_F(\mathbf{Q}')$
  Choose an element X from I_t according to $\succ_{\mathcal{I}}$
  Compute that weight $\mathbf{v}_t \in \Delta_{(m)_2}$, which maximizes $D(\mathbf{v}_t, \hat{\mathbf{Q}}_{t-1}, \mathcal{Q}_m^h(\neg \mathrm{X}))$
  Compute the projection $\mathbf{v}_t^{\varepsilon_t}$ of $\mathbf{v}_t$ onto $\Delta_{(m)_2}^{\varepsilon_t}$
  Pull $(i,j) = \mathrm{argmin}_{(i',j')\in(m)_2} (\boldsymbol{n}_t)_{i',j'} - \sum_{s=1}^{t} (\mathbf{v}_s^{\varepsilon_s})_{i',j'}$, observe $X_{i,j} \sim \mathrm{Ber}(q_{i,j})$
  Update $\boldsymbol{w}_t$ via $(\boldsymbol{w}_t)_{k,l} \leftarrow (\boldsymbol{w}_{t-1})_{k,l} + \mathbf{1}_{\{\{k,l\}=\{i,j\} \text{ and } X_{k,l}=1\}}$ $\quad \forall 1 \le k,l \le m$
  Update $\boldsymbol{n}_t$ via $(\boldsymbol{n}_t)_{k,l} \leftarrow (\boldsymbol{n}_{t-1})_{k,l} + \mathbf{1}_{\{\{k,l\}=\{i,j\}\}}$ $\quad \forall 1 \le k,l \le m$
  Update $\hat{\mathbf{Q}}_t \leftarrow \frac{\boldsymbol{w}_t}{\boldsymbol{n}_t}$.
  Let $\mathcal{D}_t \leftarrow \{\mathbf{Q}' \in \mathcal{Q}_m^h : D(\boldsymbol{n}_t/t, \hat{\mathbf{Q}}_t, \mathbf{Q}') \le \beta(t,\gamma)\}$
  if $\exists \mathrm{X} \in \mathcal{I}$ with $\mathcal{D}_t \cap \mathcal{Q}_m^h(\neg \mathrm{X}) = \emptyset$ **then**
    return X
  Update $t \leftarrow t+1$

of a difficult min-max problem; doing this for any $\mathbf{Q} \in \mathcal{C}_t$ is seemingly infeasible. This indicates that Algorithm 8 is computationally even far more complex than Algorithm 7.

To analyze its theoretical performance, recall that we have defined above for fixed $m \in \mathbb{N}$, $h \in (0,1/2)$, $\mathrm{X} \in \mathcal{I}$ and any $\mathbf{Q} \in \mathcal{Q}_m(\mathrm{X})$ the value

$$D_m^h(\mathbf{Q}) \coloneqq \sup_{\mathbf{v}\in\Delta_{(m)_2}} D(\mathbf{v}, \mathbf{Q}, \mathcal{Q}_m^h(\neg\mathrm{X})) = \sup_{\mathbf{v}\in\Delta_{(m)_2}} \inf_{\mathbf{Q}'\in\mathcal{Q}_m^h(\neg\mathrm{X})} \sum_{(r,s)\in(m)_2} v_{r,s} d_{\mathrm{KL}}(q_{r,s}, q'_{r,s}).$$

As $\mathcal{Q}_m^h = \bigcup_{\mathrm{X}\in\mathcal{I}} \mathcal{Q}_m^h(\mathrm{X})$ is a disjoint union, $D_m^h(\mathbf{Q})$ is well-defined for any $\mathbf{Q} \in \mathcal{Q}_m^h$. Similarly as in Lemma J.3 we obtain the following result. Therein, the term $m-1$ is in accordance to our lower bound from Theorem 4.1.

Lemma J.5. *(i) For any $\mathbf{Q} \in \mathcal{Q}_m^h(\mathrm{CW})$ we have*

$$D_m^h(\mathbf{Q}) \le \frac{d_h(\mathbf{Q})}{m-1}$$

with $d_h(\mathbf{Q}) \coloneqq \max_{(i,j)\in(m)_2} \max\{d_{\mathrm{KL}}(q_{i,j}, 1/2+h), d_{\mathrm{KL}}(q_{i,j}, 1/2-h)\}$.

(ii) For any $\mathbf{Q} \in \mathcal{Q}_m^h$ we have

$$D_m^h(\mathbf{Q}) \ge \frac{8h^2}{m}$$

and in case $\mathbf{Q} \in \mathcal{Q}_m^h(\mathrm{CW})$ we even obtain $D_m^h(\mathbf{Q}) \ge \frac{8h^2}{m-1}$.

Proof. (i) Let $\mathbf{Q} \in \mathcal{Q}_m^h(\mathrm{CW})$ and $\mathbf{v} \in \Delta_{(m)_2}$ be fixed for the moment. Let $i \coloneqq \mathrm{CW}(\mathbf{Q})$. By assumption on $\mathbf{v}$ there exists some $j \in [m] \setminus \{i\}$ with $v_{i,j} \le (m-1)^{-1}$. For arbitrary small but fixed $\delta \in (0, 1/2-h)$ define $\mathbf{Q}' \in \mathcal{Q}_m^h$ via

$$q'_{r,s} \coloneqq \begin{cases} 1/2 - (h+\delta), & \text{if } (r,s) = (i,j), \\ 1/2 + h + \delta, & \text{if } (r,s) = (j,i), \\ q_{r,s}, & \text{otherwise,} \end{cases}$$

for each $(r,s) \in (m)_2$. Then, $\mathbf{Q}' \in \mathcal{Q}_m^h(\neg i)$ due to $q'_{i,j} < 1/2 - h$. As $q_{i,j} > 1/2 + h$ holds by assumption on $\mathbf{Q}$ and i, the definition of $\mathbf{Q}'$ assures

$$\sum_{(r,s)\in(m)_2} v_{r,s} d_{\mathrm{KL}}(q_{r,s}, q'_{r,s}) \le v_{i,j} d_{\mathrm{KL}}(q_{i,j}, q'_{i,j}) \le \frac{d_{\mathrm{KL}}(q_{i,j}, 1/2-(h+\delta))}{m-1}.$$

Regarding $\mathbf{Q}' \in \mathcal{Q}_m^h(\neg i)$ and that this estimate is obtained for any $\mathbf{v} \in \Delta_{(m)_2}$, we can conclude that

$$D_m^h(\mathbf{Q}) = \sup_{\mathbf{v}\in\Delta_{(m)_2}} \inf_{\mathbf{Q}\in\mathcal{Q}_m^h(\neg i)} \sum_{(r,s)\in(m)_2} v_{r,s} d_{\mathrm{KL}}(q_{r,s}, q'_{r,s}) \le v_{i,j} d_{\mathrm{KL}}(q_{i,j}, q'_{i,j}) \le \frac{d_{\mathrm{KL}}(q_{i,j}, 1/2-(h+\delta))}{m-1}$$

and taking the limit $\delta \searrow 0$ yields

$$D_m^h(\mathbf{Q}) \le \frac{d_{\mathrm{KL}}(q_{i,j}, 1/2-h)}{m-1} \le \frac{d_h(\mathbf{Q})}{m-1}.$$

(ii) Suppose $\mathrm{X} \in \mathcal{I}$ and $\mathbf{Q} \in \mathcal{Q}_m^h(\mathrm{X})$ to be arbitrary. In case $\mathrm{X} = \neg\mathrm{CW}$ we have due to $\mathcal{Q}_m(\neg(\neg\mathrm{CW})) = \mathcal{Q}_m(\mathrm{CW})$ the equality $\tilde{D}_m^h(\mathbf{Q}) = D_m^h(\mathbf{Q})$, whence $D_m^h(\mathbf{Q}) \ge \frac{8h^2}{m}$ follows from Lemma J.3.
Now, consider the case $\mathrm{X} = i \in \{1,\ldots,m\}$. Define $\mathbf{v} = (v_{r,s})_{(r,s)\in(m)_2} \in \Delta_{(m)_2}$ via $v_{r,s} = \mathbf{1}_{\{i\in\{r,s\}\}}(m-1)^{-1}$ for each $(r,s) \in (m)_2$. For any $\mathbf{Q}' \in \mathcal{Q}_m^h(\neg i)$ there exists some $j \in [m] \setminus \{i\}$ with $q'_{i,j} < 1/2 - h$, as otherwise $\mathrm{CW}(\mathbf{Q}') = i$ would hold. But by assumption on i, $q_{i,j} > 1/2 + h$ holds, whence we can estimate with Lemma J.1 that

$$\sum\nolimits_{(r,s)\in(m)_2} v_{r,s} d_{\mathrm{KL}}(q_{r,s}, q'_{r,s}) \ge v_{i,j} d_{\mathrm{KL}}(q_{i,j}, q'_{i,j}) \ge \frac{d_{\mathrm{KL}}(1/2+h, 1/2-h)}{m-1} \ge \frac{8h^2}{m-1}.$$

As this holds for arbitrary $\mathbf{Q}' \in \mathcal{Q}_m^h(\neg i)$, we obtain $D_m^h(\mathbf{Q}) \ge \frac{8h^2}{m-1} > \frac{8h^2}{m}$.

□

With this, we obtain the following result, which is an analogue to Proposition J.4.

Proposition J.6. *(i) Let $h \in (0,1/2)$ and $m \in \mathbb{N}$ be fixed. Any algorithm $\mathcal{A}$, which is able to to solve the CW-testification task on $\mathcal{Q}_m^h$ for any γ (write $\mathcal{A}(\gamma)$) fulfills*

$$\liminf_{\gamma\to 0} \frac{\mathbb{E}_{\mathbf{Q}}[T^{\mathcal{A}(\gamma)}]}{\ln(\gamma^{-1})} \ge \frac{1}{D_m^h(\mathbf{Q})}.$$

In particular,

$$\sup_{\mathbf{Q}\in\mathcal{Q}_m^h(\mathrm{CW})} \liminf_{\gamma\to 0} \frac{\mathbb{E}_{\mathbf{Q}}[T^{\mathcal{A}(\gamma)}]}{\ln(\gamma^{-1})} \ge \frac{(m-1)(1/4-h^2)}{4h^2}.$$

(ii) Let $h \in (0,1/2)$ and $m \in \mathbb{N}$ be fixed. Choose $C > 0$, $(\varepsilon_t)_{t\in\mathbb{N}}$, $t \mapsto f(t)$ and $(t,\gamma) \mapsto \beta(t,\gamma)$ as in Proposition J.4. Write $\mathcal{A}(\gamma)$ for Algorithm 8 called with parameters $\gamma, h, (\varepsilon_t)_t, f$ and β. Then, $\mathcal{A}(\gamma)$ solves the CW-testification problem on $\mathcal{Q}_m^h$ for γ, and fulfills

$$\lim_{\gamma\to 0} \frac{\mathbb{E}_{\mathbf{Q}}[T^{\mathcal{A}(\gamma)}]}{\ln(\gamma^{-1})} = \frac{1}{D_m^h(\mathbf{Q})}$$

for any $\mathbf{Q} \in \mathcal{Q}_m^h$. In particular,

$$\sup\nolimits_{\mathbf{Q}\in\mathcal{Q}_m^h} \lim\nolimits_{\gamma\to 0} \frac{\mathbb{E}_{\mathbf{Q}}[T^{\mathcal{A}(\gamma)}]}{\ln(\gamma^{-1})} \le \frac{m}{8h^2}.$$

Proof. (i) Theorem 1 in [Degenne and Koolen, 2019] implies the first statement. For the second statement let $\mathbf{Q} = \mathbf{Q}(\delta) \in \mathcal{Q}_m^h$ be such that $q_{i,j} \in \{1/2 \pm (h+\delta)\}$ for all $(i,j) \in (m)_2$ and some arbitrarily small $\delta > 0$. Then,

$$\begin{aligned} d_h(\mathbf{Q}) &= \max\nolimits_{(i,j)\in(m)_2} \max\{d_{\mathrm{KL}}(q_{i,j}, 1/2+h), d_{\mathrm{KL}}(q_{i,j}, 1/2-h)\} \\ &= d_{\mathrm{KL}}(1/2+(h+\delta), 1/2-h) \le \frac{4(h+\delta/2)^2}{1/4-h^2} \end{aligned} \tag{10}$$

where we have used Lemma J.1 in the last step. Thus, the second statement follows from part (i) of Lemma J.5 by taking the limit $\delta \searrow 0$.

(ii) Theorem 11 in [Degenne and Koolen, 2019] implies the first statement. For the choice of ε_t confer p.7 in [Garivier and Kaufmann, 2016], for $f(t)$ see Lemma 14 on p.9 in [Degenne and Koolen, 2019] and for $\beta(t,\gamma)$ see Theorem 10 on p.8 in [Degenne and Koolen, 2019]. The second statement follows directly from the bound on $D_m^h(\mathbf{Q})$ stated in part (ii) of Lemma J.5.

□

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