Bachelor Thesis

Comparison of different approaches to solving sequential decision problems with applications to poker

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Abstract

This work focuses on analyzing how do different methods of solving sequential decision problems can compare to each other when trying to solve simplified versions of the game of poker. Mainly two common methods of solving a sequential decision problem are the normal form method and the extensive form method. Both methods need some sort of optimality criteria to be able to define preference orders, so in total there were seven optimality criteria chosen to make the comparison, which were the Maximax criterion, Maximin criterion, Hurwicz criterion, Laplace criterion, Minimax Regret criterion, the Bernoulli principle and the Hodges and Lehmann criterion.

A couple of differences between the usage of the same criterion with the two different methods were able to be identified and talked about in the poker problem, while still trying to understand how different scenarios can be advantageous for a certain method in comparison to the other. Another point discussed was how the different optimality criteria would be influenced by not only the methods but also different situations such as outlier value reward leaves.

Contents

Chapter 1

Introduction

What is a decision? According to the Cambridge dictionary a decision is: a choice that you make about something after thinking about several possibilities (Cambridge, n.d.). But if one thinks about real life situations, one does not necessarily have to think to make decisions. How much do we actually think when brushing our teeth or using deodorant right after waking up as it has already become a routine in our lives? One can argue that decisions are choices every living being must take which leads to some sort of action or inaction, not necessarily thought about. From simple amoeboid organism solving different problems (Reid et al., 2016), to which prey will a lion go after, to whether we will play in the lottery in the end of the year, decisions are natural occurrences of the nature of living beings.

Since the first developments of statistics as a subject, games of chance have been of utter importance and curiosity for mathematicians. Bernoulli describing the toss of a coin has had a significant impact in probability theory and how we see the entire dynamics of these sorts of games. While Bernoulli was trying to describe mathematically the game itself, other areas which have been of great interest are the different decision possibilities, as well as different strategies used. While John von Neumann and Oskar Morgenstern would develop the basis of what we know nowadays as utility theory and game theory, L. J Savage would have his work focused on utility theory and decisions itself. What would be later referenced as normative theory of decision (Morgenstern & Von Neumann, 1953; Luce & Raiffa, 1957, chapter 1).

While decision theory might be segmented into different areas, such as descriptive, normative and prescriptive decision theory, this bachelor thesis will only deal with normative decision theory. It can be defined as behavioral rules which deal with how decisions should be taken to maximize the well-being of the different decision makers.(Bacci & Chiandotto, 2019, chapter 1)

Expanding on what Von Neumann, Morgenstern, Savage and other mathematicians and researchers developed in decision theory, the idea of this bachelor thesis is to examine decision theory, more specifically sequential decision theory. Having as main objective to use all the classical and Bayesian sequential decision tools to describe a real-life example which has always interested me personally: poker.

Before going deeper into specific decision theory details in Chapter 2, there are some important introductory concepts that need to be laid out. In the decision theory field, there are some different classifications for problems, for example whether a decision is made by an individual or a group, as well as if it is affected by conditions of certainty, risk or uncertainty. (Luce & Raiffa, 1957, chapter 2)

This bachelor thesis will deal with individual decisions made under risk or uncertainty, in the sense that they appear in the work of Luce and Raiffa. As defined in games and decisions, an individual decision is a decision made by a single decision maker, either a human being or an organization, which can be seen as a collective with a unified purpose and no conflicting directions within the individuals that form it. While decisions under risk will have a set of specific outcomes for which the probabilities are known by our decision maker, decisions made under uncertainty will relate to whether an action of one party or even both will lead to a set of end results, for which the probabilities are unknown or not even meaningful. (Luce & Raiffa, 1957, chapter 2)

The structure of this thesis was chosen to highlight the theoretic nature of the topic in chapters two and three. In the second chapter the decision problem will be explained, as well as how will it be structured with mathematical notation, also establishing some important concepts and talking about the different approaches seen in the literature to solve single decision problems. In the third chapter it will be further investigated the structure of sequential decision, more specifically the decision trees format and the different possible solution approaches to them. Building on which approaches we saw for non-sequential decision problems in chapter 2 and using much of what was established in the previous chapter.

The fourth chapter will use the developments of the second and third to try to firstly solve analytically Kuhn Poker, which is a simplified version of poker, with the normal and the extensive methods. After that an expanded Kuhn Poker version will be explored, where there will 5 cards being played instead of the original Kuhn Poker with three. Further, it will be discussed what the possible different approaches from Kuhn Poker and expanded Kuhn Poker might mean to a Texas hold ‘em game. How we could possibly transfer knowledge. Lastly in the fifth chapter some conclusions will be drawn. It will be discussed where these different problem approaches might be useful in different real-life situations and not only games per say.

Chapter 2

Decision Theory

Throughout this chapter the focus will be on solving single decision problems with different approaches. In section 2.1 it will be explained the overall structure of a decision problem, while also introducing some important concepts, such as acts, states, utilities, together with the mathematical notation that will be used. Section 2.2 has as main objective going through different classical solution approaches and explaining different criteria, such as the minimax, minimax regret, maximin, Hurwicz and principle of insufficient reason. Similarly, Section 2.3 will also present how to solve the single decision problems but with the Bernoulli approach. The last chapter will explain other possible approaches, which are common in decision theory, but won't necessarily be the focal point of this project.

2.1 Structuring Decision Problems

When talking about decision problems one of the main difficulties is to structure it in a way, which enables the analysis of the different possibilities, the occurrences of the decision and be able to define possible strategies. The first important aspect of this process is naming the single entity, which will make the decision, as the decision maker.

To better illustrate such a problem, Savage presents us with an example that can help lay down the main structures of a decision problem. The example tries to explain the dilemma of a decision maker, who is trying to make a six-egg omelet. His wife has already broken five eggs into a bowl and now he must decide between three possible decisions. The first being if he will break the sixth egg in the bowl with the five other eggs, if he will break it in a separate saucer or lastly if he will completely discard it without even breaking it. The problem here is that there is always the possibility that an egg is rotten and the only way to discover it is opening it. Therefore, if the sixth egg is rotten and it is broken in the bowl, all the eggs will be wasted. While if it is broken in a saucer, the decision maker will have one more utensil to wash afterwards. If he simply decides to discard it, the decision maker might just be wasting a good egg. (Savage, 1954).

In this entire problem, there are basically three possible decisions for the decision maker, which will be from now on called acts. The two different conditions of the sixth egg (rotten or good) will be named the state of nature. Lastly the results of a decision being made together with the discovery of the true state of nature will induce a consequence. All the above-mentioned concepts are commonly seen in literature around decision theory (Luce & Raiffa, 1957; Berger, 1985, chapter 1 and chapter 1) and will be better developed later in this section. Still, for the sake of this example, it is important to already have the nomenclature for the following elucidative table around the decision makers dilemma:

Acts	States	
	Rotten	Good
Break into bowl	No omelet, and five good eggs destroyed	Six-egg omelet
Break into saucer	Five-egg omelet and a saucer to wash	Six-egg omelet and a saucer to wash
Throw away	Five-egg omelet	Five-egg omelet, and one good egg waster

Table 2.1: Savage's Dilemma towards breaking the sixth egg

The table illustrates well how one decision together with all possible states of nature will give us different consequences, but what kind of action should the decision maker take in this case? Mostly the best strategy for this decision maker depends on what kind of personal preferences he has. For example, if he absolutely hates washing the dishes, he might not choose breaking the egg into a saucer, if he doesn't mind the dishes and doesn't want the risk of spoiling five other eggs, he might just choose to break it in a saucer or throw it away entirely. There is a relative "value" to the different possible consequences. Even when there are clear monetary payoffs as consequences, a scale measuring the "value" of the consequences might not be as obvious as only using the monetary gains. An example for that is when one decision maker has no money and must do an unpleasant choir to earn 200€, while another decision maker is a billionaire and can also earn 200€ doing the same unpleasant choir. In this case, 200€ might have a completely different significance depending on the decision maker's financial situation.

To be able to work mathematically with these "values", numbers indicating such values need to be assessed. These numbers will basically be called utilities and utility theory deals exactly with how these numbers are established. (Berger, 1985, chapter 2). Although how to get these utilities and utility theory have a vast literature, every detail will neither be thoroughly explained, nor be the main focus of this bachelor thesis.

Continuing this chapter, some basic notation needs to be established for the different important concepts already introduced. Individual actions, which can be chosen by the decision maker, will be defined as a and the set of all these actions will be $A = \{a_1, a_2, ..., a_n\}$. While the states of nature will be defined as θ (a parameter) and the set of all possible states of nature will be denoted as $\Theta = \{\theta_1, \theta_2, ..., \theta_p\}$ (a parameter space). When talking about states of nature, it is important to clear that it does not really relate to nature in itself, but it might just be anything that is out of the control of the decision maker. Examples of this might be as simple as an actual natural event, such as a hurricane, to a decision from other players, that might not necessarily follow a logical or known strategy to the decision maker. The combination of a specific (θ_i, a_j) will generate a certain utility, represented by $U(\theta_i, a_j)$ or u_{ij} . In a more general setting $U(\theta, a)$[1]. The following table 2.2 with a finite set of actions, a discrete set of states of nature and the corresponding utilities might help visualize the relation between ideas:

Acts	States				
	θ_1	...	θ_j	...	θ_p
a_1	u_{11}	...	u_{1j}	...	u_{1p}
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
a_k	u_{k1}	...	u_{kj}	...	u_{kp}
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
a_n	u_{n1}	...	u_{nj}	...	u_{np}

Table 2.2: Acts x States table

Even though through many economics papers often it is seen the usage of utilities, statistician also use loss as a parameter. The idea is that the loss will be minimized, instead of a utility being maximized. This is an advantage because expected loss is the correct measure for loss in random occurrences(Berger, 1985, chapter 2). Therefore, expected loss will be used as a criterion when relating to randomized rules, such as risks, Bayesian expected loss and Bayes risks (Berger, 1985, chapter 2). If there is an existing utility $U(\theta, a)$, the loss can be defined as:

$$L(\theta, a) = -U(\theta, a) \tag{2.1}$$

To be able to define some preferences, these following ideas need to be established. Suppose a_i and a_k are two different acts in a decision problem. Further notation seen in the works of Luce and Raiffa will be used to help us define some ideas.(Luce & Raiffa, 1957, chapter 13):

i. $a_i \sim a_k$: represents that they are equivalent acts, this means that they will have the same utility in each different state of nature.

ii. $a_i \succ a_k$: represents that a_i strong dominates a_k, this means that a_i is preferred in each different state of nature.

iii. $a_i \succeq a_k$: represents that a_i weakly dominates a_k, this means that for at least one state of nature a_i is preferred to a_k and a_i is preferred or indifferent to a_k for all the other states of nature.

In the following table 2.3, it is possible to exemplify these relations better. It is possible to see that the first action is equivalent to the fourth ($a_1 \sim a_4$), at the same time the first action strongly dominates the last action ($a_1 \succ a_6$), while the first also only weakly dominates the fifth($a_1 \succeq a_5$).

But the important question is how to choose when there is no action that is clearly dominating over all the other possible actions. This will be discussed in the following chapters, how to define preferences when there is no clear “winning” strategy.

[1]The mathematical notations are based on the works of Berger(Berger, 1985, chapter 1, chapter 2) and Luce and Raiffa(Luce & Raiffa, 1957, chapter 13)

Acts	States				
	θ_1	θ_2	θ_3	θ_4	θ_5
a_1	25	10	12	15	40
a_2	26	7	15	18	35
a_3	13	25	37	13	35
a_4	25	10	12	15	40
a_5	25	10	12	15	37
a_6	23	8	10	8	33

Table 2.3: Example of a possible finite decision problem

2.2 Classical decision theory

The classical school of decision theory will mainly deal with making decisions under uncertainty. For the sake of this bachelor, it will be mostly dealing with decisions under ignorance. This means that the states of nature and their utilities are known, but no other information about these states of nature or prior probabilities about them are known(Savage, 1951; Luce & Raiffa, 1957, chapter 13).

It is important to start restricting the number of actions that might be choosen. Due to the nature of poker as a game (which will be talked about in chapter 4), this thesis will be mostly dealing with finite single decision problems. Still, finite single decision problems might have an enormous number of possible actions, therefore reducing the number of actions is still necessary. For this reason, the "non-domination principle" is useful, as it prohibits the choice of a dominated act. The though behind it is that a decision maker will not choose an action that is completely worse than another in all other states of nature. Therefore, it is possible to discard this dominated action and only stay with those actions that aren't clearly dominated by no other. Actions that are equivalent will also be taken into account only once in classical single decision problems, as they are virtually the same(Szaniawski, 1960).

In table 2.3 it is possible to see these relations. a_1 is equivalent $a_4(a_1 \sim a_4)$, while it is also weakly dominating a_5 ($a_1 \succeq a_5$)and dominating $a_6(a_1 \succ a_6)$. Therefore, based on beforehand explained principles, it would be able to simplify table 2.3 and the result would be the following modified table 2.4.

To be able to define a preference order one of the methods is using "rational" criteria, which basically are rules that define certain preference orders depending on what is optimal according to the criterion(Luce & Raiffa, 1957, chapter 13).These are the different criteria chosen to be used and discussed further on:

Acts	States				
	θ_1	θ_2	θ_3	θ_4	θ_5
a_1	25	10	12	15	40
a_2	26	7	15	18	35
a_3	13	25	37	13	35

Table 2.4: Example of a decision problem without dominated acts or repeated acts

Maximax

As the maximax/max-max criterion goes, the idea is that the maximum utility of each action will be compared and the optimal actions are the ones with the maximum overall maximum value. This will result in the chosen a_i. It is described as being an extremely optimistic criterion.(Bacci & Chiandotto, 2019, chapter 5)

$$\max_j u_{ij} = \max_k \max_j u_{kj} \tag{2.2}$$

Maximin/Wald criterion

As the maximin criterion goes, the idea is that the minimum utility of each action will be compared and the optimal actions are the ones with the maximum overall minimum value. This will result in the chosen a_i(Wald, 1945). It is considered as a pessimistic criterion(Luce & Raiffa, 1957, chapter 13).

$$\min_j u_{ij} = \max_k \min_j u_{kj} \tag{2.3}$$

Principle of insufficient reason/Laplace criterion

As the Laplace criterion goes, the idea is that each action will be compared as the average of all the action's utilities and the optimal actions are the ones with maximum overall average. This will result in the chosen a_i.(Luce & Raiffa, 1957, chapter 13)

$$\frac{1}{p}\sum_1^p u_{ij} = \max_k \frac{1}{p}\sum_1^p u_{kj} \tag{2.4}$$

Hurwicz criterion

As the Hurwicz goes, the idea is that each action will be compared through $\alpha \min_j u_{kj} + (1-\alpha)\max_j u_{kj}$. Basically comparing together, the maximum and minimum utilities of each action. Where alpha is the "pessimism" parameter ($0 \leq \alpha \leq 1$). The optimal actions are the maximum overall results from the above formula. This will result in the chosen a_i.(Hurwicz, 1951)

$$\alpha \min_j u_{ij} + (1-\alpha)\max_j u_{ij} = \max_k[\alpha \min_j u_{kj} + (1-\alpha)\max_j u_{kj}] \tag{2.5}$$

Minimax Regret criterion

As the minimax regret criterion goes, the idea is that firstly the regret will be calculated, the regret is basically the amount that must be added to the utility for it to equal the maximum utility in their respective state of nature column. What will be compared here are the maximum risks of each action and the actions with the minimum maximum risk will be chosen. This will result in the chosen a_s.(Savage, 1951) The criterion is considered to be pessimistic.(Luce & Raiffa, 1957, chapter 13)

$$\max_j(\max_k u_{ij} - u_{ij}) = \min_k \max_j(\max_k u_{kj} - u_{ij}) \tag{2.6}$$

Bernoulli principle

The Bernoulli principle is described in the literature as using the expected utility to compare and establish a preference relation between the different possible actions. This method uses the possible utilities of each action and multiplies it by the prior probability of their corresponding state of nature. The main objective is identifying and choosing the action which has the biggest overall expected utility.[2](Rommelfanger & Eickemeier, 2013, chapter 3.2)

$$\sum_{j=1}^{p} u_{ij} p_\theta(\theta_j) = \max_k \sum_{j=1}^{p} u_{kj} p_\theta(\theta_j) \tag{2.7}$$

Hodges and Lehmann criterion

The Hodges and Lehmann criterion is a mixture of two different combinations of criteria. The first part is the Bernoulli principle together with originally the minimax criterion. Although the original paper uses risk and loss for the criteria, as in this bachelor utilities are being used here, then the Bernoulli part of the criterion will be the expected utility, while the minimax will actually be the maximin for the utility, as what will be used is the best utility out of the worst possible utilities of the action(which corresponds to the minimax of the loss).(Jr. & Lehmann, 1952)

$$\lambda_0 \sum u_{ij} p_\theta(\theta_j) + (1-\lambda_0) \min_j u_{ij} = \max_k(\lambda_0 \sum u_{kj} p_\theta(\theta_j) + (1-\lambda_0) \min_j u_{kj}) \tag{2.8}$$

2.3 Examples

The aim of tables 2.5 and 2.6 are to exemplify how the former mentioned classical criteria work in the practical sense. To show how each of them might choose a different action depending on their definition of "optimal" action. Mainly this will happen due to the different focuses of each criterion. In both tables the numbers in bold from the section criterion will be the chosen optimal action according to the criterion. In table 2.5 the criteria being showed are the Maximax, Maximin, Laplace and Hurwicz, where, after the first part of each criteria is calculated and displayed, the chosen action in all of them is their maximum value. The Maximax "optimal" action will be a_1. While for both the Wald and Laplace it will be a_3 and it is possible to observe that there are for the Hurwicz criteria, in this example, two possible "optimal" actions, a_1 and a_3. An example for the preference order of the Maximax would be: a_1 over a_3 over a_2. For the other criterion the preference of order would be analogously determined.

In table 2.6 the criterion being shown is the Minimax regret criterion, for which firstly the regrets need to be calculated, afterwards the action which has the lowest maximum is chosen. In this case it will be a_3.

Although classical criteria might be useful in many different problems, it is not possible to consider data or posterior probabilities of certain situations. Therefore, it is required different criteria such as the Bernoulli and the Hodges and Lehmann

[2]In the literature, from which the principle is cited, the author uses the E for the expected utility notation, while here it is simply the extended version with the utilities being multiplied by the states of nature with the max notation in order to make all criterion having an uniform style.

Acts	States					Criterion			
	θ_1	θ_2	θ_3	θ_4	θ_5	$\max_j u_{ij}$	$\min_j u_{ij}$	$\frac{1}{p}\sum_1^p u_{ij}$	$\alpha \min_j u_{ij} + (1-\alpha)\max_j u_{ij}$
a_1	25	10	12	15	40	**40**	10	20.4	**25**
a_2	26	7	15	18	35	35	7	20.2	21
a_3	13	25	37	14	35	37	**13**	**24.8**	**25**

Table 2.5: Maximax, Wald, Laplace and Hurwicz criteria approaches to solving a single decision problem

Acts	Sates					Regret values					Criterion
	θ_1	θ_2	θ_3	θ_4	θ_5	$26 - u_{i1}$	$25 - u_{i2}$	$37 - u_{i3}$	$18 - u_{i4}$	$40 - u_{i5}$	$\max_j$
a_1	25	10	12	15	40	1	15	25	3	0	25
a_2	26	7	15	18	35	0	18	22	0	5	22
a_3	13	25	37	14	35	13	0	0	4	5	**13**

Table 2.6: Example of a decision problem without dominated acts or repeated acts

criterion. In table 2.7 it is possible to see how the example would be solved with the Bernoulli principle. It is important to note that in the example some prior probabilities of the state of nature had to be assumed. The assumed values are $P_\theta(\theta_1) = P_\theta(\theta_2) = P_\theta(\theta_5) = 0.2$, $P_\theta(\theta_3) = 0.3$ and $P_\theta(\theta_4) = 0.1$. These prior probabilities will be multiplied with the corresponding utility and the sum of all the expected utilities of that action will define different expected utilities of each action. The action with the highest utility will be chosen, in the table 2.7 that would be action a_3.

Acts	Calculating Bernoulli principle	Bernoulli Criterion
a_1	$25 * 0.2 + 10 * 0.2 + 12 * 0.3 + 15 * 0.1 + 40 * 0.2$	20.1
a_2	$26 * 0.2 + 7 * 0.2 + 15 * 0.3 + 18 * 0.1 + 35 * 0.2$	19.9
a_3	$13 * 0.2 + 25 * 0.2 + 37 * 0.3 + 14 * 0.1 + 35 * 0.2$	**27.1**

Table 2.7: Bernoulli principle solution example

The last table of this section has the example of the Hodges and Lehmann criterion. As already described, this criterion could be portrayed as a combination of the Bernoulli and the maximin criterion. As it is seen in the table, the Bernoulli part multiplied by the parameter($\lambda = 0.5$) plus the smallest value possible of that action multiplied by one minus the parameter will represents a certain value. The action with the biggest combined value will be the choosen action, in the case of this example a_3.

Acts	$\lambda Bernoulli + (1-\lambda)Minimax$	Hodges and Lehmann
a_1	$0.5(25*0.2+10*0.2+12*0.3+15*0.1+40*0.2)+0.5(10)$	15.05
a_2	$0.5(26*0.2+7*0.2+15*0.3+18*0.1+35*0.2)+0.5(7)$	13.45
a_3	$0.5(13*0.2+25*0.2+37*0.3+14*0.1+35*0.2)+0.5(13)$	**20.05**

Table 2.8: Hodges and Lehman example with $\lambda = 0.5$

Chapter 3

Sequential Decision

3.1 Decision Trees

In the last section single decision problems were vastly discussed, but how do we solve problems that have multiple actions from our decision maker and multiple different states of nature, as these occur sequentially to define possible utilities? This will be the focus of this section. The idea of sequential decision problems will be introduced through the usage of decision trees and the two different approaches on how to solve them. Although there are many discussions about which is the best way to solve these problems, the decision tree solution with normal form and extensive form are the chosen methods for this bachelor thesis. The approach, which will be used going forward in this thesis, is based on the framework established by Huntley's and Troffaes' works.(Augustin, Coolen, De Cooman, & Troffaes, 2014; Huntley & Troffaes, 2011, 2012a, 2012b)

As it was also done in chapter two, firstly the basis structure and notation for our decision trees needs to be introduced. In this thesis, a decision tree is a graphical representation of a decision problem with more than one iteration between decision maker, the different possibilities of states of nature and utilities. The decision graph will be basically assembled by decision nodes, chance nodes and reward leaves, that will contain the corresponding utilities. Visually the decision nodes will be represented by rectangles, the chance nodes will be represented by circles and reward leaves are symbolized by hexagons. The following branches after decision nodes and chances nodes are, respectively, acts and states of natures.

An example that might be useful is of whether the decision maker will go to a party (act a_1) or if he will stay at home and listen to his radio (act a_2). In both situations the music that is playing might be good (state E_1) or might be bad (state E_2). If the decision maker decides to go to the party, his friends, that are already there, might be in the first floor (state S_1) or in the second floor (state S_2) of the club. In each floor there are two dj areas and the decision maker will need to decide whether he will go to area 1(act b_1) or area 2(act b_2). On the other hand if the decision maker decided to stay at home, he must firstly choose a radio station, he can decide between radio 1(act c_1), radio 2 (act c_2) or radio 3(act c_3). This example can be visualized in Figure 3.1.

Using Huntley's and Troffaes' notations(Huntley & Troffaes, 2011), a decision tree will be considered here as the combination of smaller trees. Therefore, if $T_1, \ldots, T_n$ are different decision trees and T is composed by these trees, in the end

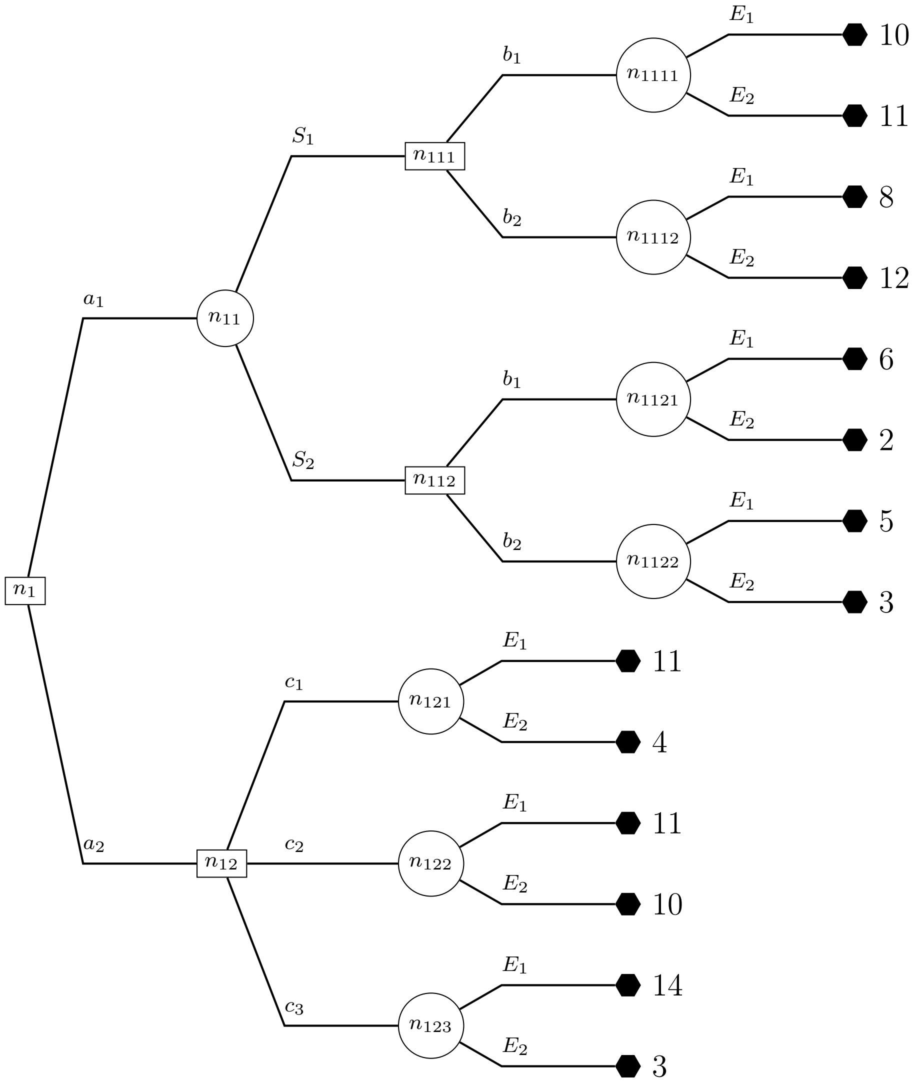

Figure 3.1: Example of a decision problem without dominated acts or repeated acts

T will be described with the usage of the notations in formulas 3.1 and 3.2:

$$T = \bigsqcup_{i=1}^{n} T_i \tag{3.1}$$

If $E_1, \ldots, E_n$ is a part of the space of the states of nature and the subtrees T_i are connected at chance nodes, by our event E_i, to form T then:

$$T = \bigodot_{i=1}^{n} E_i T_i \tag{3.2}$$

For example figure 3.1, could be described as:

$$(S_1(T_1 \sqcup T_2) \odot S_2(T_3 \sqcup T_4)) \sqcup (U_1 \sqcup U_2 \sqcup U_3) \tag{3.3}$$

With:
$T_1 = E_1 10 \odot E_2 11$ $T_2 = E_1 8 \odot E_2 12$ $T_3 = E_1 6 \odot E_2 2$
$T_4 = E_1 5 \odot E_2 3$ $U_1 = E_1 11 \odot E_2 4$ $U_2 = E_1 10 \odot E_2 11$
$U_3 = E_1 10 \odot E_2 15$

One very important aspect of our subtrees is that even though there might be subtrees, which are virtually the same (they have the same structure of nodes and arcs), these might be originated by different preceding events. For this reason, they should be treated differently. These intersection of these preceding events of the decision tree on chance nodes will be denoted as $ev(T)$.(Huntley & Troffaes, 2011)

Instances

Although the strategies for solving sequential decision problems depend strongly on the decisions of the decision maker, the utility that comes from these choices and validate these strategies are, obviously, also very dependent of the combination of the different states of nature. For this reason, it is needed a notation, which will exactly represent these possible states of nature. In this case, Ω will be the possibility space, which represent the set of all possible states of nature. The elements of Ω will be portrayed as ω, while the different subsets of Ω will be the events. [1]

The Ω permits us to have the instance $X : \Omega \to U$, which is a function that defines the utility that will be the outcome of the set of different states of nature. When $\omega \in \Omega$ is observed, then $X(\omega)$ will yield a utility value. Huntley and Troffaes define this idea as a gamble(Huntley & Troffaes, 2011), but due to the main example of this bachelor thesis being a game of poker, the word gamble might be used later in a different context, which might cause confusion.

Choice functions

In the approach that will be followed in this thesis, which is based on Huntley's and Troffaes' works(Huntley & Troffaes, 2011) with a few modifications, both the normal form and extensive form solutions to the decision trees use a choice function

[1] The entire idea of possibility spaces, events and their respective notation comes directly from Huntley's and Troffaes works(Huntley & Troffaes, 2011). These will be very useful concepts, mostly when solving sequential decision problems with the normal form.

opt. The idea behind this function is that it will compare the different instances, mainly through the different criteria established in section 2.3, and then find the corresponding optimal normal or extensive decision solution. The definition to the choice function will be taken directly from Huntley's and Troffaes' defition 5(Huntley & Troffaes, 2011)

Definition 5. *A choice function opt maps, for any non-empty event A, each non- empty finite set X of instances to a non-empty subset of this set:*

$$\emptyset \neq opt(X|A) \subseteq X \tag{3.4}$$

3.2 Normal Form

A normal form solution can be summarized as a kind of representation, where every decision must be previously stated before any action or state of nature has even occurred(Augustin et al., 2014, chapter 8). A good analogy for these kinds of solutions are recipes, which should be thoroughly followed by the decision maker. Another important property is that each decision node must specify only one decision arc to choose from. Two possible normal form solutions for the preceding decision tree example can be seen in Figure 3.2. In the first possible solution would choose to stay at home and listen to radio(act a_2) and then afterwards choose radio 1(act c_1). The other possible solution would be choose going to the club (act a_1), then if your friends are in the first floor (state S_1) go to the area 1 (act b_1) and if your friends are in the second floor (state S_2) go to are 2 (act b_2).

As in Huntley and Troffaes paper, here as well the set of all the different combinations of normal form decisions for a decision tree will be denoted as nfd(T) and a normal form solution will simply be a subset of nfd(T)(Huntley & Troffaes, 2011). Normally solutions with normal form decisions involve sorting all the different possible combinations of actions and choosing the combination, which will be considered the optimal normal form decisions(Huntley & Troffaes, 2012a). To be able to find this optimal normal form solution from all of these possible normal form decisions, a **normal form operator** is needed. The normal form operator will be a function, which is able to map each decision tree to a normal form solution. Even though for traditional normal form solution the expected utility is used to find this optimal decision, in this case, the optimal decision will not only be defined by expected utility but also be defined by some sort of optimal criteria.(Huntley & Troffaes, 2011)

Normal Form instances

As already explained, the two components that decide the outcome of the decision trees are: the decisions taken by the decision maker and the different states of nature that might occur, which were established as the instances. Therefore, for every possible normal form decision, there is a corresponding normal form instance, that will together generate the utility of the decision process. The set of all instances will be denoted as Inst(T).

Using our preceding decision tree as an example, if the chance node N_{11} and decision node N_{12} are choosen and then two paths are decided for these nodes, there will be two corresponding instances. At the chance node N_{11}, for both states S_1 and S_2, there are the two possible normal form decisions: b_1 or b_2. While for the decision

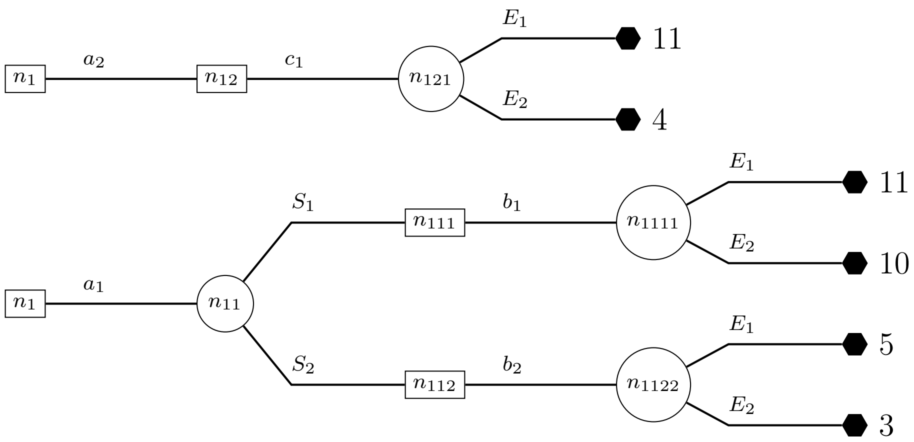

Figure 3.2: Example of two possible normal form solutions

node N_{12} there are three possible normal form decisions: c_1, c_2 or c_3. If the decision maker at N_{11} decides for b_1 in case of S_1 and b_2 if S_2 and at N_{12} for c_2, then we will have the following normal form instances:

$$S_1(E_1 10 \oplus E_2 11) \oplus S_2(E_1 5 \oplus E_2 3)$$

$$E_1 10 \oplus E_2 9$$

The $\oplus$ operator will signalize the integration of two partial maps defined on disjoint domains. The multiplication with a state of nature will mean restriction. In the first example the partial map $E_1 10$ is a constant map restricted on E_1 and $E_2 11$ is a constant map restricted on E_2, while the partial map $S_1(E_1 10 \oplus E_2 11)$ a constant map restricted on S_1 and the rest can be analogously described.

The instances permit that we tabulate these outcome possibilities, which will be very useful to compare the different normal form decisions and instances. The creation of a table is possible through the usage of the possibility space $\Omega = \{\omega_1, \omega_2, \omega_3, \omega_4\}$, as well as denoting $E_1 = \{\omega_1, \omega_2\}$ and $S_1 = \{\omega_1, \omega_3\}$. If there is no S_1 partial map in the instance, then it simply corresponds to $E_1 = \{\omega_1, \omega_2\}$ and $E_2 = \{\omega_3, \omega_4\}$. This can be seen in the following table:

Instances	ω_1	ω_2	ω_3	ω_4
$S_1(E_1 10 \oplus E_2 11) \oplus S_2(E_1 5 \oplus E_2 3)$	10	5	11	3
$E_1 10 \oplus E_2 9$	10	10	9	9

Table 3.1: Normal form instances example tabulated

The inner brackets of the first normal form instance is also another smaller instance in itself and it is possible to observe how there is a recursive component to the different instances. There are two very important recursive definitions(definition 2 and 3) from Huntley and Troffaes(Huntley & Troffaes, 2011), that will be used for the instances operator:

Definition 2. *For any events $E_1, ..., E_n$ which form a partition, and any finite family of sets of instances $X_1, ..., X_n$, we define the following set of instances:*

$$\bigoplus_{i=1}^{n} E_i X_i = \left\{ \bigoplus_{i=1}^{n} E_i X_i : X_i \in X \right\} \tag{3.5}$$

Definition 3. *With any decision tree T, we associate a set of instances $inst(T)$, recursively defined through:*

- If T consists of only a leaf with utility $u \in U$, then

$$inst(T) = \{u\} \tag{3.6}$$

- If T has a chance node as root, that is, $T = \bigodot_{i=1}^{n} E_i T_i$, then

$$inst\left(\bigodot_{i=1}^{n} E_i T_i\right) = \bigoplus_{i=1}^{n} E_i inst(T_i) \tag{3.7}$$

- If T has a decision node as root, that is, if $T = \bigsqcup_{i=1}^{n} T_i$, then

$$inst\left(\bigsqcup_{i=1}^{n} T_i\right) = \bigcup_{i=1}^{n} inst(T_i) \tag{3.8}$$

Notation for normal decision solving

Further notation will describe how our normal form operator can induce our normal form solution through a choice function. In this case definition 6 and theorem 7 from Huntley's and Troffaes' paper(Huntley & Troffaes, 2011) will be used. The idea is that as there are normal form decisions, it is possible to treat these as a single decision problem through the comparisons of the different instances. The optimal instance can be mapped to the decisions and this will be our normal form solution.

As they are being treated as a single decision problem, it enables us to use the optimality criteria introduced in section 2.2. The biggest difference here is that the instead of having separate states of nature, the idea of the possibility space will be used. Mainly this will be of great importance because not necessarily every decision path has the same state of nature, which will influence it. Therefore, an operator that makes it possible to tabulate without damages to the comparison will be very useful.

Definition 6. *Given any choice function opt, and any decision tree T with $ev(T) \neq \emptyset$, we define*

$$norm_{opt}(T) = \{D \in nfd(D) : inst(D) \subseteq opt(inst(T)|ev(T))\} \tag{3.9}$$

Theorem 7 will relate to strategic equivalence, if two subtrees have the same instance configuration, they will both have the same solution. This is a very useful property, because when there are bigger trees, it is often seen that some subtrees have exactly the same instance configuration and one does not need to solve the same problem multiple times.

Theorem 7. *If T_1 and T_2 are strategically equivalent and $ev(T_1) = ev(T_2) \neq \emptyset$, then $inst(norm_{opt}(T_1)) = inst(norm_{opt}(T_2))$*

3.2.1 Example of solving within the normal form

Although there will be seven criteria, which this bachelor thesis is interested for the next chapter, in this example only two of them will be used. This is the case because although the seven criteria have their own specificities, the form by which they are used will not necessarily dramatically change, as they are not deciding our method of problem solution, but rather giving us a definition of optimality that will be useful for the method. The main difference for the normal form will be in the case of Bernoulli principle. This difference will be explained further on.

As the notation has already been established, now it is possible to approach the practical part of actually solving different decision trees. The first normal form solution step would be finding all possible normal form decision paths and then sorting out their instances. A possibility space must be defined, in our upcoming example the possibility space $\Omega = \{\omega_1, \omega_2, \omega_3, \omega_4\}$ will be adopted, as well as denoting $E_1 = \{\omega_1, \omega_2\}$ and $S_1 = \{\omega_1, \omega_3\}$. After this step, it is possible to tabulate the instances as it was done in the previously introduced table 3.1. With the defined instances and their corresponding utility with the possibility space, it is possible to use the different classical criteria to sort which is the optimal instance and its corresponding decision. The last step would be delete any arcs that do not correspond to the normal form decision, which is optimal according to the criterion. If there are more than one optimal action that can be taken, then there will be more than one possible normal form solution. These different solutions will be segregated in two, instead of having both in the same solution(which would correspond to an extensive form solution).

Table 3.2 shows exactly all the normal form decisions together with their instances and possible utilities.

Decisions	Instances	ω_1	ω_2	ω_3	ω_4
a_1, if S_1 then b_1, if S_2 then b_1	$S_1(E_1 10 \oplus E_2 11) \oplus S_2(E_1 6 \oplus E_2 2)$	10	6	11	2
a_1, if S_1 then b_2, if S_2 then b_1	$S_1(E_1 8 \oplus E_2 12) \oplus S_2(E_1 6 \oplus E_2 2)$	8	6	12	2
a_1, if S_1 then b_1, if S_2 then b_2	$S_1(E_1 10 \oplus E_2 11) \oplus S_2(E_1 5 \oplus E_2 3$	10	5	11	3
a_1, if S_1 then b_2, if S_2 then b_2	$S_1(E_1 8 \oplus E_2 12) \oplus S_2(E_1 5 \oplus E_2 3)$	8	5	12	3
a_2 then c_1	$E_1 11 \oplus E_2 4$	11	11	4	4
a_2 then c_2	$E_1 10 \oplus E_2 9$	10	10	9	9
a_2 then c_3	$E_1 14 \oplus E_2 3$	14	14	3	3

Table 3.2: All normal form instances for our decision tree example 3.1

In table 3.2 the minimax criterion is being used to define which is the optimal instance. In this case, after getting the minimun of each action and the maximun between all the minimun utilites to each action, the optimal instance would $E_1 10 \oplus E_2 9$. The corresponding decision can be seen from table 3.2 and it would be decide to stay at home(act a_2) and then decide for radio 2 (act c_2).The normal for decision tree solution with the minimax criterion can be seen in upcoming figure 3.3.

The other criterion that is going to be analysed in this example is the Bernoulli principle. The biggest motivation for this is that the criterion won't deal directly with utilities, but it will deal with the expected utility of each instance. For this reason, instead of using the framework of tables 3.2 and 3.3, table 3.4 will be a bit

Instances Id	Instances	ω_1	ω_2	ω_3	ω_4	Criteria $\min_j u_{ij}$
isnt(1)	$S_1(E_1 10 \oplus E_2 11) \oplus S_2(E_1 6 \oplus E_2 2)$	10	6	11	2	2
isnt(2)	$S_1(E_1 8 \oplus E_2 12) \oplus S_2(E_1 6 \oplus E_2 2)$	8	6	12	2	2
isnt(3)	$S_1(E_1 10 \oplus E_2 11) \oplus S_2(E_1 5 \oplus E_2 2)$	10	5	11	3	3
isnt(4)	$S_1(E_1 8 \oplus E_2 12) \oplus S_2(E_1 5 \oplus E_2 2)$	8	5	12	3	3
isnt(5)	$E_1 11 \oplus E_2 4$	11	11	4	4	4
isnt(6)	$E_1 10 \oplus E_2 9$	10	10	9	9	**9**
isnt(7)	$E_1 14 \oplus E_2 3$	14	14	3	3	3

Table 3.3: Normal form instances with maximin criteria

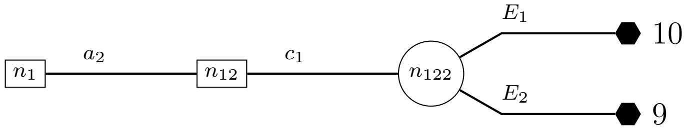

Figure 3.3: Normal form solution for the minimax criterion

different, where the middle will show how the action's expected utility is calculated. Basically the prior probabilities of the different states will be multiplied with the different rewards possible and the sum of all the combination of the strategies that form that instance will define how good is that strategy. For this example, it will be firstly arbitrarily assumed, for the sake of the example, that the state of nature S_1 has prior probability value of $P_\theta(S_1) = 0.6$ and S_2 has prior probability value of $P_\theta(S_2) = 0.4$, while E_1 has prior probability $P_\theta(E_1|S_1) = 0.5$ and E_2 has prior probability $P_\theta(E_1|S_2) = 0.5$. After these prior probabilities are assumed, it is possible to calculate the expected utility of each instance. In this case, the first instance is composed of the rewards 10 and 11, 6 and 2, this rewards will be multiplied by the prior probability of the states of nature that need to happen for it occur, so $(10P_\theta(E_1|S_1) + 11P_\theta(E_2|S_1))P_\theta(S_1) + (6P_\theta(E_1|S_2) + 2P_\theta(E_2|S_2))P_\theta(S_2)$. The idea is that the combination of strategies and possible instances that generate the highest expected utility is the optimal decision solution. In the example of table 3.4 it would be $E_1 10 \oplus E_2 9$.

Id	Bernoulli calculation	Criteria $\sum_{i=1}^{p} \mathbf{E}_\theta(u)$
ins(1)	$(10 * 0.5 + 11 * 0.5) * 0.6 + (6 * .5 + 2 * .5) * 0.4$	7.9
ins(2)	(8*0.5+12*0.5)*0.6+(6*.5+2*0.5)*0.4	7.6
ins(3)	(10*0.5+11*0.5)*0.6+(5*0.5+2*0.5)*0.4	7.7
ins(4)	(8*0.5+12*0.5)*0.6+(5*0.5+2*0.5)*0.4	7.4
ins(5)	(11*0.5+4*0.5)	7.5
ins(6)	(10*0.5+9*0.5)	**9.5**
ins(7)	(14*0.5+3*0.5)	8.5

Table 3.4: Normal form Bernoulli calculation

3.3 Extensive Form

In Raiffa and Schlaifer works(Schlaifer & Raiffa, 1961), the definition of extensive form is well established with backward induction and expected utility. As the name already suggests, the entire idea of this method is to start from the last decision node and solve it in parts, as if each partition of the bigger tree was a unique single decision problem. After solving the last decision nodes, then the penultimate decision nodes will be solved and so on and so forth, until the entire tree is solved and there is an extensive form solution.

Expanding this definition for situations in which there is no fixed utility, Huntley and Troffaes will talk about another property of the extensive form, which is that the decision arc must only be specified as it is achieved by our decision maker(Huntley & Troffaes, 2011). This will be a direct difference to the normal form and will also be part of the definition for the extensive form here. It is an important property because if there are acts that are both optimal according to the criterion used, then there is no need to choose between one of them or have two different possible solutions as it would be the case for the normal form.

Another important aspect of the extensive form is that the extensive form solution might also induce normal form solution(mainly because of how they are defined). If the extensive form solution form a solution where every decision node is followed by a single act, then it might also be called a normal form solution. Another possibility is if an extensive form has at a specific act two possible arcs, this solution can be simply segregated in to two different normal form solutions. For this reason, it is possible to understand there are an equal or greater amount of normal form solutions.

Notation for extensive form decision solving

To enable the usage of the backwards induction method as a form of getting the extensive form solution, then extra definitions will be needed together with small modifications to the original method introduced by Huntley and Troffaes. First, the definitions 23 and 24 from Huntley's and Troffaes' works(Huntley & Troffaes, 2011) will need to be introduced as they will be of immense importance to the extensive form used here.

Definition 23. *Given a choice function opt and any set T of consistent decision trees, where $ev(T) = A$ for all $T_i \in T$*

$$norm_{opt}(T) = \{D \in nfd(T) : inst(D) \subseteq opt(inst(T)|A)\} \tag{3.10}$$

Definition 24. *The normal form operator $back_{opt}$ is defined for any consistent decision tree T through:*

- If T consists of only a leaf with utility $u \in U$, then

$$back_{opt}(T) = \{T\} \tag{3.11}$$

- If T has a chance node as root, that is, $T = \bigodot_{i=1}^{n} E_i T_i$, then

$$back_{opt}\left(\bigodot_{i=1}^{n} E_i T_i\right) = norm_{opt}\left(\bigodot_{i=1}^{n} E_i back_{opt}(T_i)\right) \tag{3.12}$$

- If T has a decision node as root, that is, if $T = \bigsqcup_{i=1}^{n} T_i$, then

$$back_{opt}\left(\bigsqcup_{i=1}^{n} T_i\right) = norm_{opt}\left(\bigsqcup_{i=1}^{n} back_{opt}(T_i)\right) \tag{3.13}$$

The idea of this method is to find normal form solutions to every possible subtree, eliminating from the decision tree non optimal paths and eliminating the corresponding instances. Definition 23 is exactly denoting the application of the normal form opt to sets of the tree. While the definition 24 introduces the back opt notation, which denotes solving with the norm opt solution every stage of the tree.

The main difference to the approach in this thesis and the algorithm above is that in this thesis, even though there will be normal form optimal solutions chosen at each stage, if the same decision node has two possible normal form optimal solutions at that specific decision node, both of them will be situated in the decision tree solution. This will give us the extensive form solution through the usage of the back opt and norm opt at each part of the tree. To be able to continue solving the tree, then both of the paths will be used separately with the normal form opt, instead of one being chosen over the other.

It is also of vital importance to differentiate between normal form solutions and normal form optimal solutions, mainly because an extensive form solution might also induce a normal form solution but not necessarily a normal form optimal solution. If all the decision nodes of the extensive form solution have only one action possible after the opt choice function was used with the backward induction method, then the solution is also considered a normal form solution. Still, this does not necessarily mean that if the normal form method of solving the decision tree were used the normal form solutions given by it would be the same as the one induced by the extensive form. Therefore, normal form optimal solutions will simply be the optimal solutions induced by the normal form method, while normal forms solutions are any solutions with just one possible action chosen after every decision node. For this reason, it is also possible to get one extensive form that has just one decision node with two possible optimal decision and every other decision with one and segregate it into two different normal form solutions. Being able to translate the extensive form into multiple normal form solutions will be very important later on, mostly when comparing the usage of different criteria with normal form and extensive form methods.

3.3.1 Example of solving within the extensive form

As it was also done for the normal form section, in this subsection the focus will be put into actually solving the decision tree with the extensive form. The two optimality criteria chosen to ne used as example will also be the minimax and the Bernoulli principle. The entire idea of the backward induction is to solve the last subtree, which can be solved as a single decision problem, then after that is solved, one can solve the penultimate subtree as a single decision problem and so on and so forth, until the entire tree is solved:

These ideas can be exactly seen through Figures 3.4, 3.5 and table 3.5. Starting by the maximin example, the criteria gets the minimum of the states of nature and then chooses the maximum out of it. This can exactly be seen in figure 3.4. For the

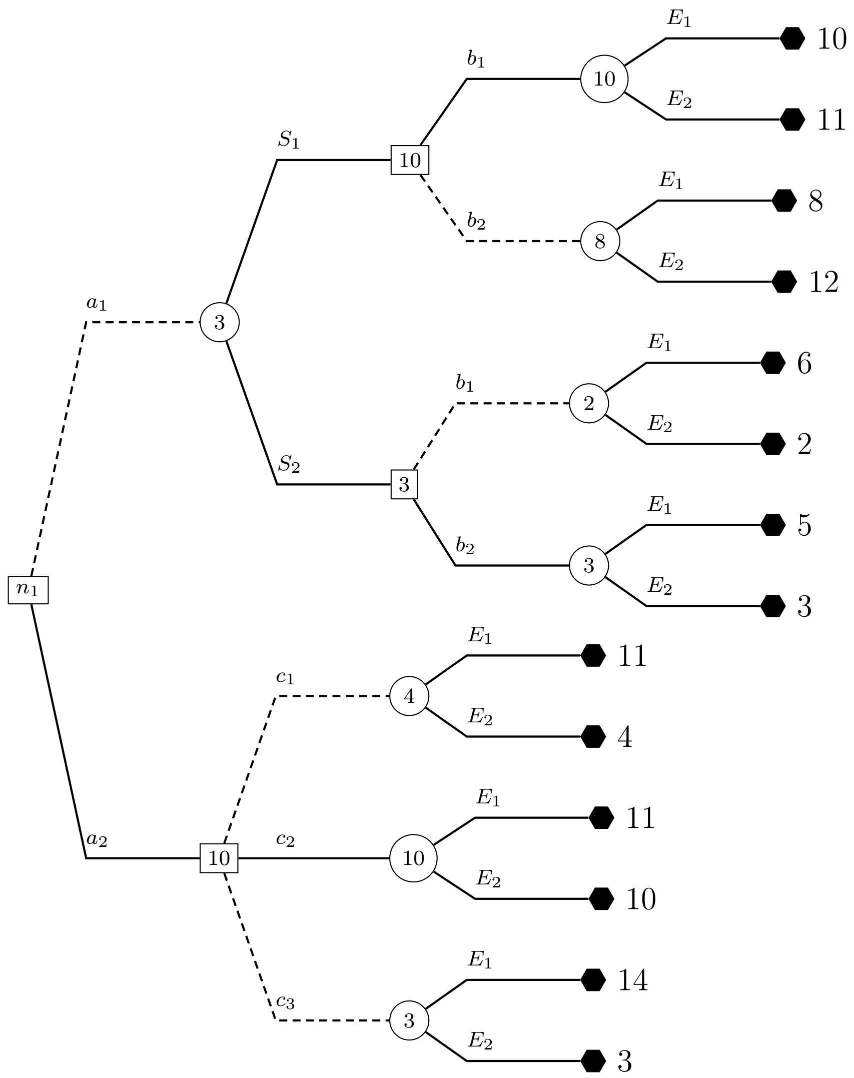

Figure 3.4: Example of a Maximin criterion with the backward induction

upper b_1 and b_2 decisions of node n_{111}, the b_1 path has as rewards leaves E_1 10 and E_2 11 and b_2 has as rewards leaves E_1 8 and E_2 12. The minimum between these rewards for b_1 is 10, which for a practical reason will be put as the node n_{1111}, while for b_2 the rewards leaves E_1 and E_2 have a minimun of 8, so the node n_{1112} will be 8. After that, the decision maker chooses the biggest possible minimun, which is obviously 10, then n_{111} will "receive" that value again and after every different problem at this height of the tree is solved, then it goes one step backwards and the decision maker can finally choose between a_1 and a_2.

In the second example it is quite a similar process with the biggest difference being that instead of using the minimum of each chance node with the rewards leaves, the method uses the expected utility of that chance node and then the decision maker decides for the path with the maximum expected utility.

Acts	States		Criterion
	E_1	E_2	$\min_j u_{ij}$
b_1	10	11	**10**
b_2	8	12	8

Table 3.5: Solving node n_{111} with maximin criteria

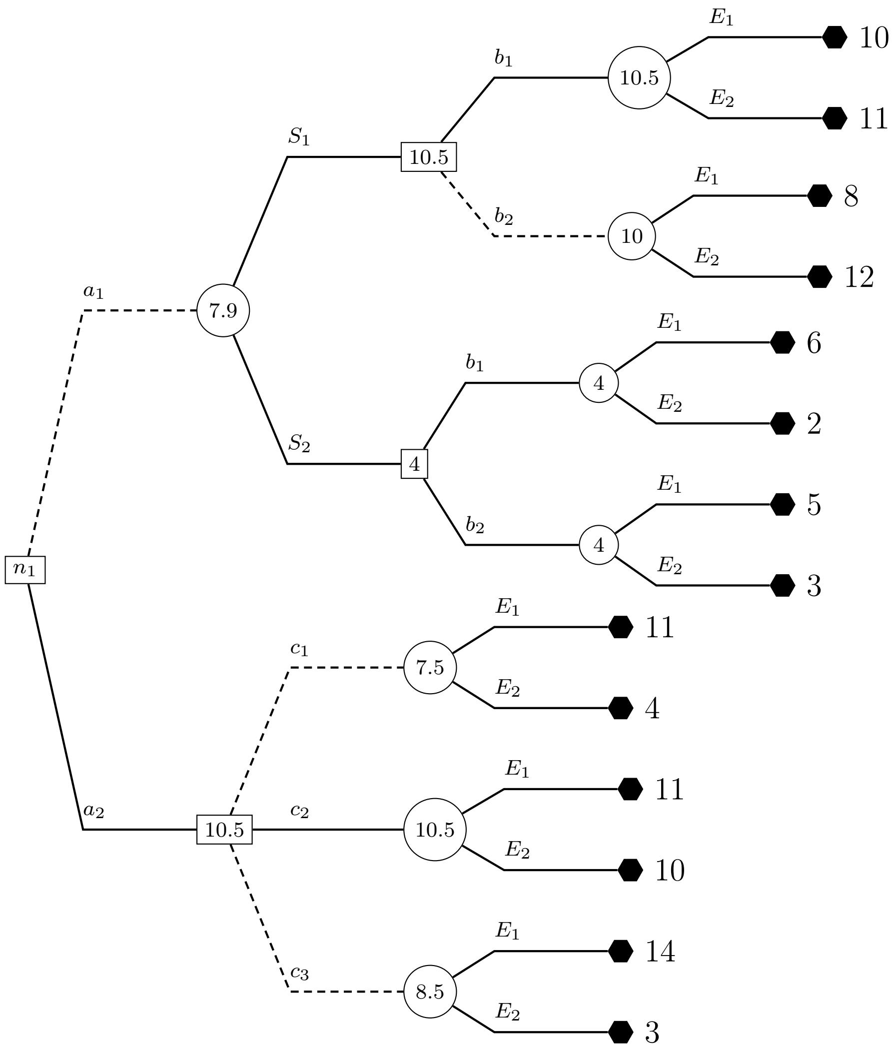

Figure 3.5: Example of a Bernoulli principle with the backward induction

Chapter 4

Sequential Decision Theory Applied to Poker

In this section it will be discussed how can poker be solved with the different sequential decision approaches that were beforehand introduced. As a game, Texas hold 'em poker, the most played and known version of poker, can be quite complex. The number of combinations together with how the cards are drawn makes it a game with quite a few rules, which elevates the difficulty of describing and solving it with decision theoretic approach, at least in this bachelor level. For this reason, it was decided that two different modified versions of poker would be used as examples. Even though these versions have less cards and are considerably simpler, the essence of the game, that means using bluffs to be able to get a better payout, will still be very much present. The main objective of this section is to be able to firstly compare the different approaches and optimality criterion with themselves and afterwards try to talk about how these different approaches might transcend to a Texas hold 'em game or if it would transcend at all.

4.1 Kuhn poker

The first version of poker that will be used as an example is called Kuhn Poker. It was created in 1950 by the mathematician Harold W. Kuhn. The idea is that only two players are facing one another and they are playing with only three cards (jack, queen and king). Each of them must put one coin down to be able to play and they will each receive a card. After evaluating their cards, the first player can decide to either check or raise, adding one more coin to the pot. In case he checks, the other player has also the option to check or raise. If then the second player raises, the first player would be able to fold or check again. In case the first player folded the other would automatically win and in case he checked, then the cards would be revealed and the player with the highest card would earn the entire pot. If the first player decided to raise instead of check, then the second player would only have the option to fold or check. If the second player folded, he would just automatically lose one coin, but if he checked, the cards would be revealed as in the last example(Kuhn, 2016).

The game was structured like this mainly for it to be simpler but also for it to be a zero-sum game, that could more easily be analyzed for game-theoretic purposes.

In this bachelor thesis it is not necessary that the game used be a zero-sum game. Therefore, some modifications will be done to the original Kuhn poker. The purpose of these modifications is to make the game more similar to poker, even though at first the three-card deck will be maintained. The first change will be that there is always the possibility to fold, as it normally is in Poker, even if it doesn't really make much sense strategically. The second main modification is that both players have the option to raise once and this doesn't depend on the action of the other player. If the second player decides to raise, then the first will be asked again what his action will be and then he has the possibility again to fold or check. Each raise still means an addition of one coin to the pot.

First, some notation will have to be given to this decision problem to be able to solve it. The notation is based on the previously introduced ideas from chapter 3 about subtrees, instances and etc. In the first occurrence of this decision process, the decision maker can receive a king, a queen or a jack. These will be the first states of nature and they will be denoted as: A_{king}, A_{queen} and A_{jack}. The following node in this problem will be a decision node, where the decision maker will have to decide if he will either raise, check or fold. This will be denoted as: b_{raise}, b_{check} and b_{fold}. The next node will be a chance node, which will portray the decision from the opponent of the decision maker. This will be written as: B_{raise}, B_{check} and B_{fold}. Although in some games another player wouldn't be considered as a state of nature, in this problem this is the case because players are, more often than not, bluffing and not playing rationally the entire game. After our opponent has decided, there are different occurrences that might happen. Independent of the first state of nature, if the opponent decides to raise after a raise or a check from the decision maker, then the decision maker will have to make another decision. This decision node will be composed of two possible decisions, whether he will check or fold. These two decisions will be portrayed by c_{check} or c_{raise}. In case the first state of nature is a queen (A_{queen}), if the decision maker decides to check after a raise from his opponent, there is still one more state of nature that will decide the outcome. This state of nature will be if the card from the opponent is a king or a jack. This will be noted as: D_{king} and D_{jack}. The following decision tree 4.1 introduces visually the problem and notation previously described.

The first important aspect that needs to be highlighted are the dashed lines in our decision tree 4.1. These were allocated for the decision c_{check} in case the first state of nature is A_{jack} and c_{fold} in case the first state of nature is A_{king}. As it was also used in the last chapter, the dashed lines will symbolize that a decision has been made and that this dashed line decisions are not part of the wanted/chosen path. There are mainly two reasons why this has already been done for these four decision nodes. The first is that all of these decisions are made under certainty. In both occasions(with A_{king} and A_{jack}) the decision maker already knows if he will win or lose, as there are only three cards in the deck. The second motive is that the objective of the decision maker is to maximize his utility if there are clear dominant strategies and in both cases one act strongly dominates over the other. For the state of nature A_{king}, it will be seen $c_{check} \succ c_{fold}$ and for A_{jack}, $c_{fold} \succ c_{check}$. Therefore, B_{raise} for these two specific states of nature (A_{king} and A_{jack}) will in the practical sense be a reward leave, with the utility of the non-dashed line decision. It will also be treated as just a reward leave when defining the notation for the subtrees and instances.

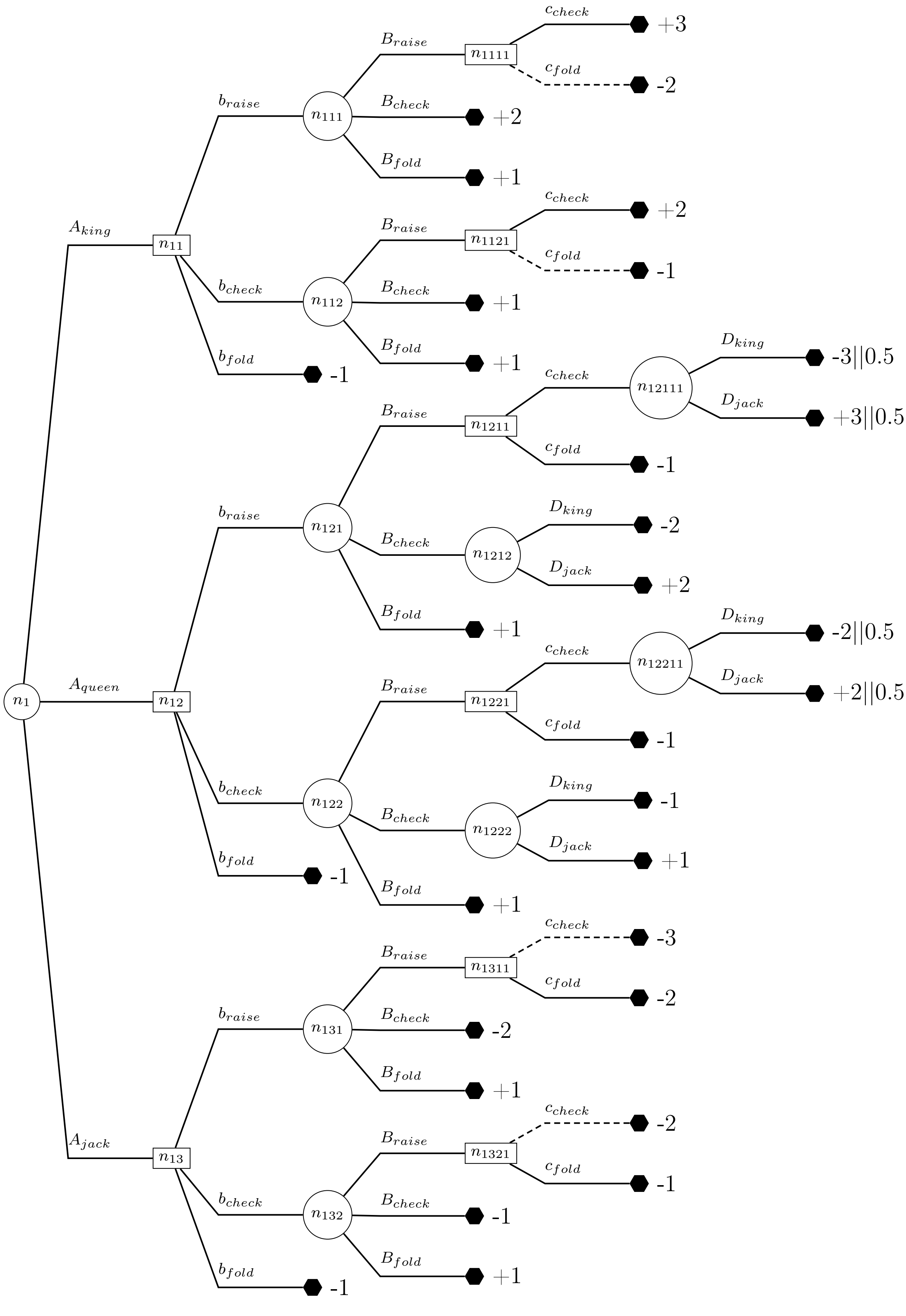

Figure 4.1: Kuhn poker decision problem with small modifications

Subtrees

As it was also done in the last section, here the Kuhn poker decision tree will be described as the combination of its subtrees. This is quite useful for a couple of different reasons. The first one is that the subtree notation facilitates the process of getting all the instances, through the direct visualization of the different possibilities of the tree, together with a closer notation to the one used for instances. The second is that it is easier to deal with solving the different subtrees with the backward induction.

$$A_{king}(T_1 \sqcup T_2 \sqcup T_3) \odot A_{queen}(T_4 \sqcup T_5 \sqcup T_3) \odot A_{jack}(T_6 \sqcup T_7 \sqcup T_3) \tag{4.1}$$

With:
$T_1 = B_{raise}3 \odot B_{check}2 \odot B_{fold}1 \quad T_2 = B_{raise}2 \odot B_{check}1 \odot B_{fold}1 \quad T_3 = -1$
$T_4 = B_{raise}(R_1 \sqcup T_3) \odot B_{check}(R_2) \odot B_{fold}1$
$T_5 = B_{raise}(R_2 \sqcup T_3) \odot B_{check}(R_3) \odot B_{fold}1$
$T_6 = B_{raise}(-2) \odot B_{check}(-2) \odot B_{fold}1 \qquad T_7 = B_{raise}(-1) \odot B_{check}(-1) \odot B_{fold}1$
And:
$R_1 = D_{king}(-3) \odot D_{jack}3 \quad R_2 = D_{king}(-2) \odot D_{jack}2 \quad R_3 = D_{king}(-1) \odot D_{jack}1$

Decisions and Instances

In this subsection both the different possible decisions, as well all the corresponding instances, will be laid out in table 4.1 and 4.2, respectively. There are some differences between how these are going to presented in chapter 4 in comparison to chapter 3. These differences are mostly due to practicality reasons, mainly because the modified Kuhn poker decision tree is a bit more complex than the example in chapter 3.

One of these modifications is that, although the subtrees may represent different decision possibilities and multiple instances, whenever a subtree represents only a certain specific instance, instead of two possibilities of strategic paths, then it will be used as the notation instead of deriving the long representation for the instance. For example, instead of writing $A_{king}(B_{raise}3 \oplus B_{check}2 \oplus B_{fold}1)$ for a possible instance, it will be simply written $A_{king}(T_1)$. This is not possible in the case of T_4 and T_5, because there are two possible instances for each of them. Another important aspect is that ,for example, T_1 won't represent $B_{raise}3 \odot B_{check}2 \odot B_{fold}1$, but whether $B_{raise}3 \oplus B_{check}2 \oplus B_{fold}1$. This small difference in notation has to be beforehand made explicit, for there to be no confusion, as the instances' different utilities will be partially mapped by others.

The second important aspect is that as both the decisions and the instances are quite long pieces of information, it will be quite a big task to write the full corresponding decision or instance whenever the normal form is being solved with a certain criterion. Therefore, each decision and instance will be referenced through a certain identification symbol, which will be η_i and i is the index for the decision or instance. It makes sense to use one only one symbol for both, because, as it was already explained last chapter, a decision has it's corresponding instance.

Id	Decisions
η_1	If A_{king} then b_{raise}; if A_{queen} then b_{raise} and if B_{raise} then c_{check}; if A_{jack} then b_{raise}
η_2	If A_{king} then b_{raise}; if A_{queen} then b_{raise} and if B_{raise} then c_{fold}; if A_{jack} then b_{raise}
η_3	If A_{king} then b_{raise}; if A_{queen} then b_{check} and if B_{raise} then c_{check}; if A_{jack} then b_{raise}
η_4	If A_{king} then b_{raise}; if A_{queen} then b_{check} and if B_{raise} then c_{fold}; if A_{jack} then b_{raise}
η_5	If A_{king} then b_{raise}; if A_{queen} then b_{fold}; if A_{jack} then b_{raise}
η_6	If A_{king} then b_{raise}; if A_{queen} then b_{raise} and if B_{raise} then c_{check} ; if A_{jack} then b_{check}
η_7	If A_{king} then b_{raise}; if A_{queen} then b_{raise} and if B_{raise} then c_{fold} ; if A_{jack} then b_{check}
η_8	If A_{king} then b_{raise}; if A_{queen} then b_{check} and if B_{raise} then c_{check} ; if A_{jack} then b_{check}
η_9	If A_{king} then b_{raise}; if A_{queen} then b_{check} and if B_{raise} then c_{fold} ; if A_{jack} then b_{check}
η_{10}	If A_{king} then b_{raise}; if A_{queen} then b_{fold} ; if A_{jack} then b_{check}
η_{11}	If A_{king} then b_{raise}; if A_{queen} then b_{raise} and if B_{raise} then c_{check} ; if A_{jack} then b_{fold}
η_{12}	If A_{king} then b_{raise}; if A_{queen} then b_{raise} and if B_{raise} then c_{fold}; if A_{jack} then b_{fold}
η_{13}	If A_{king} then b_{raise}; if A_{queen} then b_{check} and if B_{raise} then c_{check} ; if A_{jack} then b_{fold}
η_{14}	If A_{king} then b_{raise}; if A_{queen} then b_{check} and if B_{raise} then c_{fold}; if A_{jack} then b_{fold}
η_{15}	If A_{king} then b_{raise}; if A_{queen} then b_{fold}; if A_{jack} then b_{fold}
η_{16}	If A_{king} then b_{check}; if A_{queen} then b_{raise} and if B_{raise} then c_{check}; if A_{jack} then b_{raise}
η_{17}	If A_{king} then b_{check}; if A_{queen} then b_{raise} and if B_{raise} then c_{fold}; if A_{jack} then b_{raise}
η_{18}	If A_{king} then b_{check}; if A_{queen} then b_{check} and if B_{raise} then c_{check} ; if A_{jack} then b_{raise}
η_{19}	If A_{king} then b_{check}; if A_{queen} then b_{check} and if B_{raise} then c_{fold}; if A_{jack} then b_{raise}
η_{20}	If A_{king} then b_{check}; if A_{queen} then b_{fold}; if A_{jack} then b_{raise}
η_{21}	If A_{king} then b_{check}; if A_{queen} then b_{raise} and if B_{raise} then c_{check} ; if A_{jack} then b_{check}
η_{22}	If A_{king} then b_{check}; if A_{queen} then b_{raise} and if B_{raise} then c_{fold} ; if A_{jack} then b_{check}
η_{23}	If A_{king} then b_{check}; if A_{queen} then b_{check} and if B_{raise} then c_{check} ; if A_{jack} then b_{check}
η_{24}	If A_{king} then b_{check}; if A_{queen} then b_{check} and if B_{raise} then c_{fold} ; if A_{jack} then b_{check}
η_{25}	If A_{king} then b_{check}; if A_{queen} then b_{fold} ; if A_{jack} then b_{check}
η_{26}	If A_{king} then b_{check}; if A_{queen} then b_{raise} and if B_{raise} then c_{check}; if A_{jack} then b_{fold}
η_{27}	If A_{king} then b_{check}; if A_{queen} then b_{raise} and if B_{raise} then c_{fold}; if A_{jack} then b_{fold}
η_{28}	If A_{king} then b_{check}; if A_{queen} then b_{check} and if B_{raise} then c_{check} ; if A_{jack} then b_{fold}
η_{29}	If A_{king} then b_{check}; if A_{queen} then b_{check} and if B_{raise} then c_{fold}; if A_{jack} then b_{fold}
η_{30}	If A_{king} then b_{check}; if A_{queen} then b_{fold} ; if A_{jack} then b_{fold}
η_{31}	If A_{king} then b_{fold}; if A_{queen} then b_{raise} and if B_{raise} then c_{check}; if A_{jack} then b_{raise}
η_{32}	If A_{king} then b_{fold}; if A_{queen} then b_{raise} and if B_{raise} then c_{fold} ; if A_{jack} then b_{raise}
η_{33}	If A_{king} then b_{fold}; if A_{queen} then b_{check} and if B_{raise} then c_{check}; if A_{jack} then b_{raise}
η_{34}	If A_{king} then b_{fold}; if A_{queen} then b_{check} and if B_{raise} then c_{fold}; if A_{jack} then b_{raise}
η_{35}	If A_{king} then b_{fold}; if A_{queen} then b_{fold}; if A_{jack} then b_{raise}
η_{36}	If A_{king} then b_{fold}; if A_{queen} then b_{raise} and if B_{raise} then c_{check} ; if A_{jack} then b_{check}
η_{37}	If A_{king} then b_{fold}; if A_{queen} then b_{raise} and if B_{raise} then c_{fold} ; if A_{jack} then b_{check}
η_{38}	If A_{king} then b_{fold}; if A_{queen} then b_{check} and if B_{raise} then c_{check} ; if A_{jack} then b_{check}
η_{39}	If A_{king} then b_{fold}; if A_{queen} then b_{check} and if B_{raise} then c_{fold} ; if A_{jack} then b_{check}
η_{40}	If A_{king} then b_{fold}; if A_{queen} then b_{fold} ; if A_{jack} then b_{check}
η_{41}	If A_{king} then b_{fold}; if A_{queen} then b_{raise} and if B_{raise} then c_{check} ; if A_{jack} then b_{fold}
η_{42}	If A_{king} then b_{fold}; if A_{queen} then b_{raise} and if B_{raise} then c_{fold}; if A_{jack} then b_{fold}
η_{43}	If A_{king} then b_{fold}; if A_{queen} then b_{check} and if B_{raise} then c_{check} ; if A_{jack} then b_{fold}
η_{44}	If A_{king} then b_{fold}; if A_{queen} then b_{check} and if B_{raise} then c_{fold}; if A_{jack} then b_{fold}
η_{45}	If A_{king} then b_{fold} ; if A_{queen} then b_{fold} ; if A_{jack} then b_{fold}

Table 4.1: All the combination of normal form decision possible for the poker decision tree example 4.1

Decision number	Instances
η_1	$A_{king}(T_1) \oplus A_{queen}(B_{raise}(R_1) \oplus B_{check}(R_2) \oplus B_{fold}1) \oplus A_{jack}(T_6)$
η_2	$A_{king}(T_1) \oplus A_{queen}(B_{raise}(T_3) \oplus B_{check}(R_2) \oplus B_{fold}1) \oplus A_{jack}(T_6)$
η_3	$A_{king}(T_1) \oplus A_{queen}(B_{raise}(R_2) \odot B_{check}(R_3) \odot B_{fold}1) \oplus A_{jack}(T_6)$
η_4	$A_{king}(T_1) \oplus A_{queen}(B_{raise}(T_3) \odot B_{check}(R_3) \odot B_{fold}1) \oplus A_{jack}(T_6)$
η_5	$A_{king}(T_1) \oplus A_{queen}(T_3) \oplus A_{jack}(T_6)$
η_6	$A_{king}(T_1) \oplus A_{queen}(B_{raise}(R_1) \oplus B_{check}(R_2) \oplus B_{fold}1) \oplus A_{jack}(T_7)$
η_7	$A_{king}(T_1) \oplus A_{queen}(B_{raise}(T_3) \oplus B_{check}(R_2) \oplus B_{fold}1) \oplus A_{jack}(T_7)$
η_8	$A_{king}(T_1) \oplus A_{queen}(B_{raise}(R_2) \odot B_{check}(R_3) \odot B_{fold}1) \oplus A_{jack}(T_7)$
η_9	$A_{king}(T_1) \oplus A_{queen}(B_{raise}(T_3) \odot B_{check}(R_3) \odot B_{fold}1) \oplus A_{jack}(T_7)$
η_{10}	$A_{king}(T_1) \oplus A_{queen}(T_3) \oplus A_{jack}(T_7)$
η_{11}	$A_{king}(T_1) \oplus A_{queen}(B_{raise}(R_1) \oplus B_{check}(R_2) \oplus B_{fold}1) \oplus A_{jack}(T_3)$
η_{12}	$A_{king}(T_1) \oplus A_{queen}(B_{raise}(T_3) \oplus B_{check}(R_2) \oplus B_{fold}1) \oplus A_{jack}(T_3)$
η_{13}	$A_{king}(T_1) \oplus A_{queen}(B_{raise}(R_2) \odot B_{check}(R_3) \odot B_{fold}1) \oplus A_{jack}(T_3)$
η_{14}	$A_{king}(T_1) \oplus A_{queen}(B_{raise}(T_3) \odot B_{check}(R_3) \odot B_{fold}1) \oplus A_{jack}(T_3)$
η_{15}	$A_{king}(T_1) \oplus A_{queen}(T_3) \oplus A_{jack}(T_3)$
η_{16}	$A_{king}(T_2) \oplus A_{queen}(B_{raise}(R_1) \oplus B_{check}(R_2) \oplus B_{fold}1) \oplus A_{jack}(T_6)$
η_{17}	$A_{king}(T_2) \oplus A_{queen}(B_{raise}(T_3) \oplus B_{check}(R_2) \oplus B_{fold}1) \oplus A_{jack}(T_6)$
η_{18}	$A_{king}(T_2) \oplus A_{queen}(B_{raise}(R_2) \odot B_{check}(R_3) \odot B_{fold}1) \oplus A_{jack}(T_6)$
η_{19}	$A_{king}(T_2) \oplus A_{queen}(B_{raise}(T_3) \odot B_{check}(R_3) \odot B_{fold}1) \oplus A_{jack}(T_6)$
η_{20}	$A_{king}(T_2) \oplus A_{queen}(T_3) \oplus A_{jack}(T_6)$
η_{21}	$A_{king}(T_2) \oplus A_{queen}(B_{raise}(R_1) \oplus B_{check}(R_2) \oplus B_{fold}1) \oplus A_{jack}(T_7)$
η_{22}	$A_{king}(T_2) \oplus A_{queen}(B_{raise}(T_3) \oplus B_{check}(R_2) \oplus B_{fold}1) \oplus A_{jack}(T_7)$
η_{23}	$A_{king}(T_2) \oplus A_{queen}(B_{raise}(R_2) \odot B_{check}(R_3) \odot B_{fold}1) \oplus A_{jack}(T_7)$
η_{24}	$A_{king}(T_2) \oplus A_{queen}(B_{raise}(T_3) \odot B_{check}(R_3) \odot B_{fold}1) \oplus A_{jack}(T_7)$
η_{25}	$A_{king}(T_2) \oplus A_{queen}(T_3) \oplus A_{jack}(T_7)$
η_{26}	$A_{king}(T_2) \oplus A_{queen}(B_{raise}(R_1) \oplus B_{check}(R_2) \oplus B_{fold}1) \oplus A_{jack}(T_3)$
η_{27}	$A_{king}(T_2) \oplus A_{queen}(B_{raise}(T_3) \oplus B_{check}(R_2) \oplus B_{fold}1) \oplus A_{jack}(T_3)$
η_{28}	$A_{king}(T_2) \oplus A_{queen}(B_{raise}(R_2) \odot B_{check}(R_3) \odot B_{fold}1) \oplus A_{jack}(T_3)$
η_{29}	$A_{king}(T_2) \oplus A_{queen}(B_{raise}(T_3) \odot B_{check}(R_3) \odot B_{fold}1) \oplus A_{jack}(T_3)$
η_{30}	$A_{king}(T_2) \oplus A_{queen}(T_3) \oplus A_{jack}(T_3)$
η_{31}	$A_{king}(T_3) \oplus A_{queen}(B_{raise}(R_1) \oplus B_{check}(R_2) \oplus B_{fold}1) \oplus A_{jack}(T_6)$
η_{32}	$A_{king}(T_3) \oplus A_{queen}(B_{raise}(T_3) \oplus B_{check}(R_2) \oplus B_{fold}1) \oplus A_{jack}(T_6)$
η_{33}	$A_{king}(T_3) \oplus A_{queen}(B_{raise}(R_2) \odot B_{check}(R_3) \odot B_{fold}1) \oplus A_{jack}(T_6)$
η_{34}	$A_{king}(T_3) \oplus A_{queen}(B_{raise}(T_3) \odot B_{check}(R_3) \odot B_{fold}1) \oplus A_{jack}(T_6)$
η_{35}	$A_{king}(T_3) \oplus A_{queen}(T_3) \oplus A_{jack}(T_6)$
η_{36}	$A_{king}(T_3) \oplus A_{queen}(B_{raise}(R_1) \oplus B_{check}(R_2) \oplus B_{fold}1) \oplus A_{jack}(T_7)$
η_{37}	$A_{king}(T_3) \oplus A_{queen}(B_{raise}(T_3) \oplus B_{check}(R_2) \oplus B_{fold}1) \oplus A_{jack}(T_7)$
η_{38}	$A_{king}(T_3) \oplus A_{queen}(B_{raise}(R_2) \odot B_{check}(R_3) \odot B_{fold}1) \oplus A_{jack}(T_7)$
η_{39}	$A_{king}(T_3) \oplus A_{queen}(B_{raise}(T_3) \odot B_{check}(R_3) \odot B_{fold}1) \oplus A_{jack}(T_7)$
η_{40}	$A_{king}(T_3) \oplus A_{queen}(T_3) \oplus A_{jack}(T_7)$
η_{41}	$A_{king}(T_3) \oplus A_{queen}(B_{raise}(R_1) \oplus B_{check}(R_2) \oplus B_{fold}1) \oplus A_{jack}(T_3)$
η_{42}	$A_{king}(T_3) \oplus A_{queen}(B_{raise}(T_3) \oplus B_{check}(R_2) \oplus B_{fold}1) \oplus A_{jack}(T_3)$
η_{43}	$A_{king}(T_3) \oplus A_{queen}(B_{raise}(R_2) \odot B_{check}(R_3) \odot B_{fold}1) \oplus A_{jack}(T_3)$
η_{44}	$A_{king}(T_3) \oplus A_{queen}(B_{raise}(T_3) \odot B_{check}(R_3) \odot B_{fold}1) \oplus A_{jack}(T_3)$
η_{45}	$A_{king}(T_3) \oplus A_{queen}(T_3) \oplus A_{jack}(T_3)$

Table 4.2: All normal form instances for the poker decision tree example 4.1

4.2 Solving Kuhn poker with decision tree normal form

The first step to be able to successfully use the different criterion together with the normal form, after the different instances are already found, is to define an accurate possibility space for these instances. For this normal form problem solving the possibility space will be defined by possibility space:

$$\Omega = \{\omega_1, \omega_2, \omega_3, \omega_4, \omega_5, \omega_6, \omega_7, \omega_8, \omega_9, \omega_{10}, \omega_{11}\} \tag{4.2}$$

as well as denoting:
$A_{king} = \{\omega_1, \omega_2, \omega_3\}$, $A_{queen} = \{\omega_4, \omega_5, \omega_6, \omega_7, \omega_8\}$, $A_{jack} = \{\omega_9, \omega_{10}, \omega_{11}\}$,
$B_{raise} = \{\omega_1, \omega_4, \omega_5, \omega_9\}$, $B_{check} = \{\omega_2, \omega_6, \omega_7, \omega_{10}\}$, $B_{fold} = \{\omega_3, \omega_8, \omega_{11}\}$
$D_{king} = \{\omega_4, \omega_6\}$ and $D_{jack} = \{\omega_5, \omega_7\}$

For example, if the instance η_1(which corresponds to $A_{king}(B_{raise}3 \oplus B_{check}2 \oplus B_{fold}1) \oplus A_{queen}(B_{raise}(D_{king}(-3) \oplus D_{jack}3) \oplus B_{check}(D_{king}(-2) \oplus D_{jack}2) \oplus B_{fold}1) \oplus A_{jack}(B_{raise}(-2) \oplus B_{check}(-2) \oplus B_{fold}1))$ is used as a small example, then the normal form mapping would look like:

Instance	ω_1	ω_2	ω_3	ω_4	ω_5	ω_6	ω_7	ω_8	ω_9	ω_{10}	ω_{11}
η_1	3	2	1	-3	3	-2	2	1	-2	-2	1

Maximax

It is observable through table 4.3 that there are quite a few optimal Maximax criterion normal form solutions. To be more specific, there are 21 possible normal form solutions with the Maximax criterion, which is short of half of all possible strategies. As it was already explained in part 2.3, the Maximax criterion is quite optimistic, but what is also observed is how it might be described as even more overly optimistic, while solving a sequential problem with normal form. This is the case because it might focus too much on only one specific reward leaf of one possible chance node. As our problem has three different chance nodes in the beginning (A_{king}, A_{queen} and A_{jack}), if there is a maximum which is specific to one of the paths, for example A_{king}, strategies for the other possible paths, in this case A_{queen} and A_{jack}, are completely neglected. This can be better exemplified with decision/instance η_{41}, where the Maximax(3) happens in one of the reward leaves from the path of A_{queen}. For this decision strategy, if the decision maker receives a queen he might receive the best reward possible, at the same time if he receives either king or jack he will definitely lose money. In the instances η_1, η_6 and η_{11}, even though the Maximax is seen in both A_{king} and A_{queen}, the path for A_{jack} is completely neglected.

If the Maximax criterion would be combined with some sort of prior knowledge about the chance node paths or if the different paths were segregated and then combined afterwards, the Maximax criterion would possibly be able to filter strategies that are more balanced and not so dependent on single rewards. Still, this would modify the the Maximax choice function(through the usage of prior information) or it wouldn't be the traditional definition of normal form solution, modifying the purpose of the normal form solution with Maximax choice function. For this reason, it is difficult to see how the plain normal form problem solving with the Maximax choice function can help to find useful solution for this specific problem, in a practical sense, where the main objective of the player is winning.

Maximin criterion

A similar problematic to the Maximax can also be observed in the Maximin criterion in table 4.3, but now with a pessimistic approach. Although there are way less possible normal form solutions to the modified Kuhn poker problem with the Maximin than with Maximax, it can still be observed how this optimality criterion can neglect some aspects of the decision tree. In this case, because of the characteristics of the Maximin, it might induce a somewhat overly pessimistic decision combination. It won't "allow" that any of the possible rewards are lower than a certain threshold, but if all of the rewards are very close to the Maximin and one is Maximin or are all Maximin, then the criterion might simply neglect, from a logical standpoint, better strategies. This can be better seen through the optimal solution η_{45}, where although the strategy is Maximin optimal, all of the possible rewards are negative utilities. How much would a strategy, where the player is certainly going to lose money in every scenario, actually be useful to him? If a decision maker receives a king, from a human perspective, he would probably not tend to go to a strategy(η_{45}), where he loses money with the best card in the game. Of course, there are still possible strategies that might be quite useful, for example, η_9 or η_{10}. But the conclusion for the normal form with Maximin criterion in a modified Kuhn poker setting has to be that only establishing as a condition a lower bound for rewards might just neglect better strategies from a gambling stance.

Laplace criterion

Through table 4.4 it is possible to analyse the different Laplace criterion normal form solutions. It is quite interesting how the number of solutions is significantly lower for the Laplace normal form solution in comparison to Maximax and Maximin. There are only two possible normal form solutions, which are n_6 and n_8. In the first solution the decision maker will b_{raise} with a king, while with a queen he will b_{raise} and if the opponent raises c_{check} and with a jack he will simply b_{check}. This makes a lot of sense from a heuristic point of view. When the decision maker has a king, he knows he will win, therefore it makes sense to take the strategy that will maximize his gains. When the decision maker has a queen, he might decide to go for the riskier strategy, where his payout might be bigger but his loss might also be bigger, if the opponent has a king and not a jack. In case the decision maker has a jack, then he will just try to minimize his probable loss if the game reaches the end.

In the second solution the decision maker will b_{raise} with a king, while with a queen he will b_{check}, if the opponent raises then c_{check} and with a jack he will simply b_{check}. This is quite a similar decision strategy to n_6 with the main difference being that with a queen he will check instead of raise. This is less riskier strategy but the payout is a bit smaller in case the decision maker wins with a queen. At the same time his loss is also a bit smaller if his opponent has a king. This strategy seems

Instance	ω_1	ω_2	ω_3	ω_4	ω_5	ω_6	ω_7	ω_8	ω_9	ω_{10}	ω_{11}	Criterion Maximax	Maximin
η_1	3	2	1	-3	3	-2	2	1	-2	-2	1	**3**	-3
η_2	3	2	1	-1	-1	-2	2	1	-2	-2	1	**3**	-2
η_3	3	2	1	-2	2	-1	1	1	-2	-2	1	**3**	-2
η_4	3	2	1	-1	-1	-1	1	1	-2	-2	1	**3**	-2
η_5	3	2	1	-1	-1	-1	-1	-1	-2	-2	1	**3**	-2
η_6	3	2	1	-3	3	-2	2	1	-1	-1	1	**3**	-3
η_7	3	2	1	-1	-1	-2	2	1	-1	-1	1	**3**	-2
η_8	3	2	1	-2	2	-1	1	1	-1	-1	1	**3**	-2
η_9	3	2	1	-1	-1	-1	1	1	-1	-1	1	**3**	**-1**
η_{10}	3	2	1	-1	-1	-1	-1	-1	-1	-1	1	**3**	**-1**
η_{11}	3	2	1	-3	3	-2	2	1	-1	-1	-1	**3**	-3
η_{12}	3	2	1	-1	-1	-2	2	1	-1	-1	-1	**3**	-2
η_{13}	3	2	1	-2	2	-1	1	1	-1	-1	-1	**3**	-2
η_{14}	3	2	1	-1	-1	-1	1	1	-1	-1	-1	**3**	**-1**
η_{15}	3	2	1	-1	-1	-1	-1	-1	-1	-1	-1	**3**	**-1**
η_{16}	2	1	1	-3	3	-2	2	1	-2	-2	1	**3**	-3
η_{17}	2	1	1	-1	-1	-2	2	1	-2	-2	1	2	-2
η_{18}	2	1	1	-2	2	-1	1	1	-2	-2	1	2	-2
η_{19}	2	1	1	-1	-1	-1	1	1	-2	-2	1	2	-2
η_{20}	2	1	1	-1	-1	-1	-1	-1	-2	-2	1	2	-2
η_{21}	2	1	1	-3	3	-2	2	1	-1	-1	1	**3**	-3
η_{22}	2	1	1	-1	-1	-2	2	1	-1	-1	1	2	-2
η_{23}	2	1	1	-2	2	-1	1	1	-1	-1	1	2	-2
η_{24}	2	1	1	-1	-1	-1	1	1	-1	-1	1	2	**-1**
η_{25}	2	1	1	-1	-1	-1	-1	-1	-1	-1	1	2	**-1**
η_{26}	2	1	1	-3	3	-2	2	1	-1	-1	-1	**3**	-3
η_{27}	2	1	1	-1	-1	-2	2	1	-1	-1	-1	2	-2
η_{28}	2	1	1	-2	2	-1	1	1	-1	-1	-1	2	-2
η_{29}	2	1	1	-1	-1	-1	1	1	-1	-1	-1	2	**-1**
η_{30}	2	1	1	-1	-1	-1	-1	-1	-1	-1	-1	2	**-1**
η_{31}	-1	-1	-1	-3	3	-2	2	1	-2	-2	1	**3**	-3
η_{32}	-1	-1	-1	-1	-1	-2	2	1	-2	-2	1	2	-2
η_{33}	-1	-1	-1	-2	2	-1	1	1	-2	-2	1	2	-2
η_{34}	-1	-1	-1	-1	-1	-1	1	1	-2	-2	1	1	-2
η_{35}	-1	-1	-1	-1	-1	-1	-1	-1	-2	-2	1	1	-2
η_{36}	-1	-1	-1	-3	3	-2	2	1	-1	-1	1	**3**	-3
η_{37}	-1	-1	-1	-1	-1	-2	2	1	-1	-1	1	2	-2
η_{38}	-1	-1	-1	-2	2	-1	1	1	-1	-1	1	2	-2
η_{39}	-1	-1	-1	-1	-1	-1	1	1	-1	-1	1	1	**-1**
η_{40}	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	1	1	**-1**
η_{41}	-1	-1	-1	-3	3	-2	2	1	-1	-1	-1	**3**	-3
η_{42}	-1	-1	-1	-1	-1	-2	2	1	-1	-1	-1	2	-2
η_{43}	-1	-1	-1	-2	2	-1	1	1	-1	-1	-1	2	-2
η_{44}	-1	-1	-1	-1	-1	-1	1	1	-1	-1	-1	1	**-1**
η_{45}	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	**-1**

Table 4.3: Solving the normal form with Maximax and Maximin choice function

quite reasonable, but it is important to highlight that a sensible way of playing might involve using both n_6 and n_8 solutions and not just one, mainly because if the decision maker only b_{raises} with a king, then his opponent might find him quite easy to read and predictable in his way of playing. The normal form with Laplace criterion, in comparison to the Maximax and Maximin, is better able to actually take into account all the rewards and not neglect certain parts of the modified Kuhn poker decision tree. It is possible to see how it is able in this example to maximize when the decision maker has an advantage(when he gets a king) and minimizes when the decision maker has a disadvantage(when he draws a jack), while also giving the decision maker two possibilities when there are two equally interesting options(when he draws a queen).

Hurwicz criterion

In table 4.4 the normal form solution for Hurwicz criterion with "pessimism" parameter $\lambda = 0.5$ can be analysed. There are two more solutions than the solution with the Laplace method. It is also possible to recognize the impact of having the combination of both Maximax and Maximin as a criterion in itself, through common characteristics already discussed in the respective sections talking about these two criteria. It is observable in this example that when the decision maker is playing with the king, then he will choose the path with the best possible reward, which is raising. On the other hand, if the decision maker draws a queen or a jack he is just trying to play safe, through either checking or folding and having a lower bound reward which combined will form the optimal action.

It is possible to describe this method as being somewhat balanced for the specific example of the modified Kuhn Poker, this is mainly because the values of the different reward leaves aren't diametrically different. But as the criterion is dependent of only two values, there might be cases where the entirety of the action will not necessarily be as good as the conjunction of the best combined upper and lower bound rewards. For example, there might cases where the the combination of the upper and lower bound give out the optimal solution, but all the other reward's from that action are negative(either a bit higher than the lower bound value or the same as). It is a question of how useful an action which has so many "losing" rewards leaves is while playing. There might also be similar cases to the Maximax, in the sense that outlier rewards might be so influential, that they will either influence the entire optimal action to be too optimistic or too pessimistic and although the "pessimism" parameter might correct this a bit, one single reward leaf might have a very big influence in the entirety of whether that specific action is an optimal Hurwicz solution.

Minimax Regret criterion

In table 4.5 it is possible to see the four different normal form solutions with the Minimax regret. As the reward leaves are integer values, the lowest possible integer that the Minimax regret can take is 1(if it were 0, then that action would be at least weakly dominating all the other and the other actions could be discarded). Therefore, in these solutions where the Minimax regret is 1, the rewards leaves will have either the best regret(0) or (1), which is the second best reward leaf possible

Instance	ω_1	ω_2	ω_3	ω_4	ω_5	ω_6	ω_7	ω_8	ω_9	ω_{10}	ω_{11}	Criterion Laplace	Criterion Hurwicz
η_1	3	2	1	-3	3	-2	2	1	-2	-2	1	0.36	0
η_2	3	2	1	-1	-1	-2	2	1	-2	-2	1	0.18	0.5
η_3	3	2	1	-2	2	-1	1	1	-2	-2	1	0.36	0.5
η_4	3	2	1	-1	-1	-1	1	1	-2	-2	1	0.18	0.5
η_5	3	2	1	-1	-1	-1	-1	-1	-2	-2	1	-0.18	0.5
η_6	3	2	1	-3	3	-2	2	1	-1	-1	1	**0.55**	0.5
η_7	3	2	1	-1	-1	-2	2	1	-1	-1	1	0.36	0.5
η_8	3	2	1	-2	2	-1	1	1	-1	-1	1	**0.55**	0.5
η_9	3	2	1	-1	-1	-1	1	1	-1	-1	1	0.36	**1**
η_{10}	3	2	1	-1	-1	-1	-1	-1	-1	-1	1	0	**1**
η_{11}	3	2	1	-3	3	-2	2	1	-1	-1	-1	0.36	0
η_{12}	3	2	1	-1	-1	-2	2	1	-1	-1	-1	0.18	0.5
η_{13}	3	2	1	-2	2	-1	1	1	-1	-1	-1	0.36	0.5
η_{14}	3	2	1	-1	-1	-1	1	1	-1	-1	-1	0.18	**1**
η_{15}	3	2	1	-1	-1	-1	-1	-1	-1	-1	-1	-0.18	**1**
η_{16}	2	1	1	-3	3	-2	2	1	-2	-2	1	0.18	0
η_{17}	2	1	1	-1	-1	-2	2	1	-2	-2	1	0	0
η_{18}	2	1	1	-2	2	-1	1	1	-2	-2	1	0.18	0
η_{19}	2	1	1	-1	-1	-1	1	1	-2	-2	1	0	0
η_{20}	2	1	1	-1	-1	-1	-1	-1	-2	-2	1	-0.36	0
η_{21}	2	1	1	-3	3	-2	2	1	-1	-1	1	0.36	0
η_{22}	2	1	1	-1	-1	-2	2	1	-1	-1	1	0.18	0
η_{23}	2	1	1	-2	2	-1	1	1	-1	-1	1	0.36	0
η_{24}	2	1	1	-1	-1	-1	1	1	-1	-1	1	0.18	0.5
η_{25}	2	1	1	-1	-1	-1	-1	-1	-1	-1	1	-0.18	0.5
η_{26}	2	1	1	-3	3	-2	2	1	-1	-1	-1	0.18	0
η_{27}	2	1	1	-1	-1	-2	2	1	-1	-1	-1	0	0
η_{28}	2	1	1	-2	2	-1	1	1	-1	-1	-1	0.18	0
η_{29}	2	1	1	-1	-1	-1	1	1	-1	-1	-1	0	0.5
η_{30}	2	1	1	-1	-1	-1	-1	-1	-1	-1	-1	-0.36	0.5
η_{31}	-1	-1	-1	-3	3	-2	2	1	-2	-2	1	-0.45	0
η_{32}	-1	-1	-1	-1	-1	-2	2	1	-2	-2	1	-0.64	0
η_{33}	-1	-1	-1	-2	2	-1	1	1	-2	-2	1	-0.45	0
η_{34}	-1	-1	-1	-1	-1	-1	1	1	-2	-2	1	-0.64	-0.5
η_{35}	-1	-1	-1	-1	-1	-1	-1	-1	-2	-2	1	-1	-0.5
η_{36}	-1	-1	-1	-3	3	-2	2	1	-1	-1	1	-0.27	0
η_{37}	-1	-1	-1	-1	-1	-2	2	1	-1	-1	1	-0.45	0
η_{38}	-1	-1	-1	-2	2	-1	1	1	-1	-1	1	-0.27	0
η_{39}	-1	-1	-1	-1	-1	-1	1	1	-1	-1	1	-0.45	0
η_{40}	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	1	-0.82	0
η_{41}	-1	-1	-1	-3	3	-2	2	1	-1	-1	-1	-0.45	0
η_{42}	-1	-1	-1	-1	-1	-2	2	1	-1	-1	-1	-0.64	0
η_{43}	-1	-1	-1	-2	2	-1	1	1	-1	-1	-1	-0.45	0
η_{44}	-1	-1	-1	-1	-1	-1	1	1	-1	-1	-1	-0.64	0
η_{45}	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1

Table 4.4: Solving the normal form with Laplace and Hurwicz($\lambda = 0.5$) criteria

of that possibility space reward. This means that the normal form solutions for the Minimax regret have quite constant reward leaves, which will bring out, for the modified Kuhn Poker, quite balanced strategies. For example, one of the optimal solutions was the already discussed η_8. Which maximizes our advantage(drawn king), decides for a safer strategy when there is no clear strategy(drawn queen) and minimizes the loss of money in the disadvantageous scenario(jack drawn). Because of the choices with queen, the Minimax can be described as a more conservative criterion.

Another aspect of the normal form with the Minimax regret is that one could think that the more 0's the possibility space has, the better the solution will be, but this might not necessarily be the case. If the actions, where there are more regrets with value one are overly compensated by the 0's of the regrets of that action, then one could argument that heuristically that action might be better. Still, both of these actions would be considered optimal normal form solutions with the Minimax regret. It is important to repeat that the different solutions for the Minimax could be used in a game of Kuhn Poker, as a form to confuse and not make the decision maker's way of playing obvious to his opponent.

Prior probability discussion

The Bernoulli principle differentiates itself quite a lot from other criteria with the addition of prior probabilities. This allows the decision maker to add some sort of knowledge about the different states of nature to the process of deciding and strategy drafting. In the specific case of this Kuhn poker problem, there are some clear prior probabilities that can be deduced through the problem description. For example, as it is known there are three cards in the deck at the moment the game is played, one can say that the probability of receiving a king, a queen and a jack is one third each ($P_\theta(A_{king}) = P_\theta(A_{queen}) = P_\theta(A_{jack}) = \frac{1}{3}$). Another known fact is that whenever the decision maker has a queen, the other player will only be able to have a king or a jack. Therefore, it is already known that $P_\theta(D_{king}|A_{queen} \cap B_{raise}) = P_\theta(D_{jack}|A_{queen} \cap B_{raise}) = P_\theta(D_{king}|A_{queen} \cap B_{check}) = P_\theta(D_{jack}|A_{queen} \cap B_{check}) = \frac{1}{2}$.

What is still not known is about how the opponent of the decision maker is going to play, whether he will raise, check or fold(B_{raise}, B_{check} or B_{fold}). It is also possible to discriminate the different occasions, when the opponent will raise more, check more of fold more. If the decision maker has a king, then the opponent, which has either a queen or a jack, might fold more than he would if the decision maker had a jack ($P_\theta(B_{fold}|A_{king}) > P_\theta(B_{fold}|A_{jack})$). These differences can be quantifiable.

Although these information are not known, there are two forms of getting these prior probabilities to be able to use the Bernoulli principle. The first would be simply deducing which kind of player the opponent is, through personality traits or other similar factors. For example, one could assume that a player has a balanced style of play and this would mean he will have similar numbers on how many times he raises, checks or folds. In this case, the probability of each choice after our decision maker has gotten a queen(A_{queen}) could be something near $P_\theta(\cap B_{raise}|A_{queen}) \approx P_\theta(B_{check}|A_{queen}) \approx P_\theta(B_{fold}|A_{queen})$. But using things as simple as personality traits or perceptions might be quite difficult to quantify, it might also be the case that the perceptions are simply wrong. For this reason, the second method will be a more reliable procedure.

Instance	Possibility's Space Regret											Criterion
	ω_1	ω_2	ω_3	ω_4	ω_5	ω_6	ω_7	ω_8	ω_9	ω_{10}	ω_{11}	Minimax Regret
η_1	0	0	0	2	0	1	0	0	1	1	0	2
η_2	0	0	0	0	4	1	0	0	1	1	0	4
η_3	0	0	0	1	1	0	1	0	1	1	0	**1**
η_4	0	0	0	0	4	0	1	0	1	1	0	4
η_5	0	0	0	0	4	0	3	2	1	1	0	4
η_6	0	0	0	2	0	1	0	0	0	0	0	2
η_7	0	0	0	0	4	1	0	0	0	0	0	4
η_8	0	0	0	1	1	0	1	0	0	0	0	**1**
η_9	0	0	0	0	4	0	1	0	0	0	0	4
η_{10}	0	0	0	0	4	0	3	2	0	0	0	4
η_{11}	0	0	0	2	0	1	0	0	0	0	2	2
η_{12}	0	0	0	0	4	1	0	0	0	0	2	4
η_{13}	0	0	0	1	1	0	1	0	0	0	2	2
η_{14}	0	0	0	0	4	0	1	0	0	0	2	4
η_{15}	0	0	0	0	4	0	3	2	0	0	2	4
η_{16}	1	1	0	2	0	1	0	0	1	1	0	2
η_{17}	1	1	0	0	4	1	0	0	1	1	0	4
η_{18}	1	1	0	1	1	0	1	0	1	1	0	**1**
η_{19}	1	1	0	0	4	0	1	0	1	1	0	4
η_{20}	1	1	0	0	4	0	3	2	1	1	0	4
η_{21}	1	1	0	2	0	1	0	0	0	0	0	2
η_{22}	1	1	0	0	4	1	0	0	0	0	0	4
η_{23}	1	1	0	1	1	0	1	0	0	0	0	**1**
η_{24}	1	1	0	0	4	0	1	0	0	0	0	4
η_{25}	1	1	0	0	4	0	3	2	0	0	0	4
η_{26}	1	1	0	2	0	1	0	0	0	0	2	2
η_{27}	1	1	0	0	4	1	0	0	0	0	2	4
η_{28}	1	1	0	1	1	0	1	0	0	0	2	2
η_{29}	1	1	0	0	4	0	1	0	0	0	2	4
η_{30}	1	1	0	0	4	0	3	2	0	0	2	4
η_{31}	4	3	2	2	0	1	0	0	1	1	0	4
η_{32}	4	3	2	0	4	1	0	0	1	1	0	4
η_{33}	4	3	2	1	1	0	1	0	1	1	0	4
η_{34}	4	3	2	0	4	0	1	0	1	1	0	4
η_{35}	4	3	2	0	4	0	3	2	1	1	0	4
η_{36}	4	3	2	2	0	1	0	0	0	0	0	4
η_{37}	4	3	2	0	4	1	0	0	0	0	0	4
η_{38}	4	3	2	1	1	0	1	0	0	0	0	4
η_{39}	4	3	2	0	4	0	1	0	0	0	0	4
η_{40}	4	3	2	0	4	0	3	2	0	0	0	4
η_{41}	4	3	2	2	0	1	0	0	0	0	2	4
η_{42}	4	3	2	0	4	1	0	0	0	0	2	4
η_{43}	4	3	2	1	1	0	1	0	0	0	2	4
η_{44}	4	3	2	0	4	0	1	0	0	0	2	4
η_{45}	4	3	2	0	4	0	3	2	0	0	2	4

Table 4.5: Minimax Regret

The second form of getting this prior probability would be empirically through observing how the opponent has played in the past. This past could be either all the previous game rounds the opponent has played throughout his life or just the short time span where he has been playing against the current decision maker. It might also be interesting to be able to differentiate how the opponent plays under pressure with almost no money to when he plays with no pressure and a lot of money. How these priors would be calculated in a practical sense, would be basically count how many times he made each decision dependent of the decision maker's card. This is very useful, because one might never know what he has if he simply folds before the end of the round. Therefore in a game where the decision maker has drawn seven queen cards and the opponent choose raise twice, check three times and twice to fold, then the prior for these states of nature would be $P_{\theta}(B_{raise}|A_{queen}) = 2/7$, $P_{\theta}(B_{check}|A_{queen}) = 3/7$ and $P_{\theta}(B_{fold}|A_{queen}) = 2/7$.

Bernoulli principle

Differently than the other criteria, the Bernoulli principle doesn't use the normal form instances to define which decisions are the optimal solution for the decision maker. Instead, the Bernoulli principle uses prior probability together with the utilities of the rewards leaves. As it was already mentioned in the last subsection, some of the prior probabilities are already known but still the preferences of our opponent are not yet known. To be able to develop this example, some of the probabilities need to be assumed. There will be three different possibilities for the different probabilities combination that will be tested out.

The first important aspect to highlight is that the different cases won't be discriminated , as it was described in the subsection prior probability. This means that the prior probability for B_{raise} after A_{king}, will be the same as B_{raise} after A_{queen} or A_{jack}. There is one practical argument for not discriminating between the different cases, which is whenever the decision maker is playing, it is easier for him to remember what his opponent did, if he raised, checked or fold how many times, instead of remembering how many times his opponent raised, checked or folded depending of his own cards. The e_i will be used to represent the different probabilities for the act of raising, checking or folding, where the i is the index for raise(1), check(2) or fold(3). In this case $P(B_{raise}|A_{king}) = P(B_{raise}|A_{queen}) = P(B_{raise}|A_{jack}) = e_1$ and analogously defined for the other two cases. This will facilitate a bit the example. These probabilities will also be used for the Hodges and Lehmann criterion. In the first the combination the opponent will have a more riskier strategy, where he will raise 80 percent of the time, while checking only 10 percent and folding only 10 percent as well($e_1 = 0.8; e_2 = 0.1; e_3 = 0.1$). In the second possible strategy the opponent will play a balanced strategy, where the probabilities are equal($e_1 = e_2 = e_3 = 1/3$) and in the last combination of probabilities the player will be risk a verse and he will fold 80 percent of the time, check 10 and raise 10($e_1 = 0.8; e_2 = 0.1; e_3 = 0.1$)[1]

It is important to denote that the calculations that compose the Bernoulli principle depend of the combination of the expected utility from the different branch paths of the tree. In this case, it depends of the expected utility for the strategy if

[1]the different probability combinations will be represented in the tables by the number 1, 2 and 3, respectively representing these are as they were mentioned in the text

the decision maker draws a king plus if the draws a queen plus if he draws a jack. In case he draws a king, he will have tree possible paths to take with tree possible expected utilities depending on the probability combination that is being used. If he draws a queen he will have five and if he draws a jack he will have two more. These possible parts of the general expected utility are denoted down below and will be summed up directly in table 4.6. The main reason is to allow the better visualization of what strategy is being calculated with which other strategies for the others cards.

In table 4.6 it is possible to see the three different calculations for the different forms the opponent can play. What is most interesting is that independently of how the opponent is playing the Bernoulli principle will still have the same two solutions(η_6 and η_8) for all three different probability combinations. The other important aspect that has to be highlighted is that both Laplace and Bernoulli principle have exactly the same optimal decision solutions. In the subsection, where the Laplace solutions were explained, it was also better explained why these solution might be reasonable and how one decision maker might decide to use them.

King

$$(3P_\theta(B_{raise}|A_{king}) + 2P_\theta(B_{check}|A_{king}) + 1P_\theta(B_{fold}|A_{king}))P_\theta(A_{king}) = (3e_1 + 2e_2 + 1e_3)\frac{1}{3} = 1e_1 + \frac{2}{3}e_2 + \frac{1}{3}e_3$$

$$2P_\theta(B_{raise}|A_{king}) + 1P_\theta(B_{check}|A_{king} + 1P_\theta(B_{fold}|A_{king}))P_\theta(A_{king}) = (2e_1 + 1e_2 + 1e_3)\frac{1}{3} = \frac{2}{3}e_1 + \frac{1}{3}e_2 + \frac{1}{3}e_3$$

$$-1P(A_{king}) = -1\tfrac{1}{3} = -\tfrac{1}{3}$$

Queen

$$((-3P_\theta(D_{king}|A_{queen} \cap B_{check}) + 3P_\theta(D_{jack}|A_{queen} \cap B_{check}))P_\theta(B_{raise}|A_{queen}) + (-2P_\theta(D_{king}|A_{queen} \cap B_{check}) + 2P_\theta(D_{jack}|A_{queen} \cap B_{check}))P_\theta(B_{check}|A_{queen}) + 1P_\theta(B_{fold}|A_{queen}))P_\theta(A_{queen}) = ((-3*0.5+3*0.5)e_1+(-2*0.5+2*0.5)e_2+e_3)\frac{1}{3} = \frac{1}{3}e_3$$

$$(-1P_\theta(B_{raise}|A_{queen})+(-2*P_\theta(D_{king}|A_{queen})+2*P_\theta(D_{jack}|A_{queen}))P_\theta(B_{check}|A_{queen})+ 1P_\theta(B_{fold}|A_{queen}))P_\theta(A_{queen}) = ((-1e_1+(-2*0.5+2*0.5)e_2+e_3)\frac{1}{3} = -\frac{1}{3}e_1+\frac{1}{3}e_3$$

$$((-2P_\theta(D_{king}|A_{queen} \cap B_{check}) + 2P_\theta(D_{jack}|A_{queen} \cap B_{check}))P_\theta(B_{raise}|A_{queen}) + + (-1P_\theta(D_{king}|A_{queen} \cap B_{check}) + 1P_\theta(D_{jack}|A_{queen} \cap B_{check}))P_\theta(B_{check}|A_{queen}) + 1P_\theta(B_{fold}|A_{queen}))P_\theta(A_{queen}) = ((-2*0.5+2*0.5)e_1+(-1*0.5+1*0.5)e_2+e_3)\frac{1}{3} = \frac{1}{3}e_3$$

$(-1P_\theta(B_{raise}|A_{queen})+(-1*P_\theta(D_{king}|A_{queen})+1*P_\theta(D_{jack}|A_{queen}))P_\theta(B_{check}|A_{queen})+$

$1P_\theta(B_{fold}|A_{queen}))P_\theta(A_{queen}) = ((-1e_1+(-1*0.5+1*0.5)e_2+e_3)\frac{1}{3} = -\frac{1}{3}e_1+\frac{1}{3}e_3$

$-1P_\theta(A_{king}) = -1\frac{1}{3} = -\frac{1}{3}$

Jack

$$((-2P_\theta(B_{raise}|A_{jack}) - 2P_\theta(B_{check}|A_{jack}) + 1P_\theta(B_{raise}|A_{jack})))P_\theta(A_{jack}) = (-2e_1 - 2e_2 + 1e_3)\frac{1}{3} = -\frac{2}{3}e_1 - \frac{2}{3}e_2 + \frac{1}{3}e_3$$

$$(-1P_\theta(B_{raise}|A_{jack}) - 1P_\theta(B_{check}|A_{jack}) + 1P_\theta(B_{raise}|A_{jack})))P_\theta(A_{jack}) = (-1e_1 - 1e_2 + 1e_3)\frac{1}{3} = -\frac{1}{3}e_1 - \frac{1}{3}e_2 + \frac{1}{3}e_3$$

$-1P_\theta(A_{jack}) = -1\frac{1}{3} = -\frac{1}{3}$

Hodges and Lehmann criterion

It was already described that the Hodges and Lehmann criterion is a combination of the Bernoulli principle and the Maximin criterion, but how does it differ in this example from just a Bernoulli normal form solution or a Maximin solution. Firstly, it is possible to see through tables 4.3, 4.4, 4.5, 4.6 and more specifically table 4.7 that from all of the different criteria, the Hodges and Lehmann is the only criterion that gives just one possible normal form solution to the modified Kuhn Poker problem. The solution, which corresponds to strategy η_9, is also at the same time a Maximin optimal solution but not a Bernoulli optimal solution. What is observable is that the Bernoulli optimal solutions have by themselves a pretty risky downside which is the scenario, where a queen is drawn and our opponent has a king. But through the addition of a Maximin, these two riskier solutions are then discarded and the only chosen solution is also a somewhat balanced strategy. In the first part of the strategy the decision maker chooses to b_{raise}, which has big possible rewards, while with a queen he will b_{check} and if the opponent B_{raise} then the decision maker will simply c_{fold} and with a jack he will just b_{check}. This exploits the advantageous scenario of receiving the best card in the game but plays it in a more conservative way when relating to either not so clear scenarios or disadvantageous scenarios.

Another important aspect to highlight is that other possible Maximin solutions will not be a normal form solution with the Hodges and Lehmann criterion, mainly because their general expected utility aren't as good, even thought their minimum reward leaf is the best out of all possible minimum reward leaves. For the modified Kuhn poker example, it is possible to describe the Hodges and Lehmann criterion as being balanced but at the same time a bit more conservative than the Bernoulli principle mainly due to the addition of the Maximin criterion.

Instance	Bernoulli principle calculation	Bernoulli principle		
		1	2	3
η_1	$1e_1 + \frac{2}{3}e_2 + \frac{1}{3}e_3 + \frac{1}{3}e_3 - \frac{2}{3}e_1 - \frac{2}{3}e_2 + \frac{1}{3}e_3$	0.37	0.44	0.83
η_2	$1e_1 + \frac{2}{3}e_2 + \frac{1}{3}e_3 - \frac{1}{3}e_1 + \frac{1}{3}e_3 - \frac{2}{3}e_1 - \frac{2}{3}e_2 + \frac{1}{3}e_3$	0.1	0.33	0.8
η_3	$1e_1 + \frac{2}{3}e_2 + \frac{1}{3}e_3 - \frac{2}{3}e_1 - \frac{2}{3}e_2 + \frac{1}{3}e_3$	0.37	.44	0.83
η_4	$1e_1 + \frac{2}{3}e_2 + \frac{1}{3}e_3 - \frac{1}{3}e_1 + \frac{1}{3}e_3 - \frac{2}{3}e_1 - \frac{2}{3}e_2 + \frac{1}{3}e_3$	0.1	0.33	0.80
η_5	$1e_1 + \frac{2}{3}e_2 + \frac{1}{3}e_3 - \frac{1}{3} - \frac{2}{3}e_1 - \frac{2}{3}e_2 + \frac{1}{3}e_3$	0	0	0.23
η_6	$1e_1 + \frac{2}{3}e_2 + \frac{1}{3}e_3 + \frac{1}{3}e_3 - \frac{1}{3}e_1 - \frac{1}{3}e_2 + \frac{1}{3}e_3$	**0.67**	**0.67**	**0.9**
η_7	$1e_1 + \frac{2}{3}e_2 + \frac{1}{3}e_3 - \frac{1}{3}e_1 + \frac{1}{3}e_3 - \frac{1}{3}e_1 - \frac{1}{3}e_2 + \frac{1}{3}e_3$	0.4	0.56	0.87
η_8	$1e_1 + \frac{2}{3}e_2 + \frac{1}{3}e_3 + \frac{1}{3}e_3 - \frac{1}{3}e_1 - \frac{1}{3}e_2 + \frac{1}{3}e_3$	**0.67**	**0.67**	**0.9**
η_9	$1e_1 + \frac{2}{3}e_2 + \frac{1}{3}e_3 - \frac{1}{3}e_1 + \frac{1}{3}e_3 - \frac{1}{3}e_1 - \frac{1}{3}e_2 + \frac{1}{3}e_3$	0.4	0.56	0.87
η_{10}	$1e_1 + \frac{2}{3}e_2 + \frac{1}{3}e_3 - \frac{1}{3} - \frac{1}{3}e_1 - \frac{1}{3}e_2 + \frac{1}{3}e_3$	0.3	0.22	0.33
η_{11}	$1e_1 + \frac{2}{3}e_2 + \frac{1}{3}e_3 + \frac{1}{3}e_3 - \frac{1}{3}$	0.6	0.44	0.37
η_{12}	$1e_1 + \frac{2}{3}e_2 + \frac{1}{3}e_3 - \frac{1}{3}e_1 + \frac{1}{3}e_3 - \frac{1}{3}$	0.33	0.33	0.33
η_{13}	$1e_1 + \frac{2}{3}e_2 + \frac{1}{3}e_3 + \frac{1}{3}e_3 - \frac{1}{3}$	0.6	0.44	0.37
η_{14}	$1e_1 + \frac{2}{3}e_2 + \frac{1}{3}e_3 - \frac{1}{3}e_1 + \frac{1}{3}e_3 - \frac{1}{3}$	0.33	0.33	0.33
η_{15}	$1e_1 + \frac{2}{3}e_2 + \frac{1}{3}e_3 - \frac{1}{3} - \frac{1}{3}$	0.23	0	-0.23
η_{16}	$\frac{2}{3}e_1 + \frac{1}{3}e_2 + \frac{1}{3}e_3 + \frac{1}{3}e_3 - \frac{2}{3}e_1 - \frac{2}{3}e_2 + \frac{1}{3}e_3$	0.07	0.22	0.76
η_{17}	$\frac{2}{3}e_1 + \frac{1}{3}e_2 + \frac{1}{3}e_3 - \frac{1}{3}e_1 + \frac{1}{3}e_3 - \frac{2}{3}e_1 - \frac{2}{3}e_2 + \frac{1}{3}e_3$	-0.20	0.11	0.73
η_{18}	$\frac{2}{3}e_1 + \frac{1}{3}e_2 + \frac{1}{3}e_3 + \frac{1}{3}e_3 - \frac{2}{3}e_1 - \frac{2}{3}e_2 + \frac{1}{3}e_3$	0.07	0.22	0.76
η_{19}	$\frac{2}{3}e_1 + \frac{1}{3}e_2 + \frac{1}{3}e_3 - \frac{1}{3}e_1 + \frac{1}{3}e_3 - \frac{2}{3}e_1 - \frac{2}{3}e_2 + \frac{1}{3}e_3$	-0.20	0.11	0.73
η_{20}	$\frac{2}{3}e_1 + \frac{1}{3}e_2 + \frac{1}{3}e_3 - \frac{1}{3} - \frac{2}{3}e_1 - \frac{2}{3}e_2 + \frac{1}{3}e_3 + \frac{1}{3}e_1 + \frac{1}{3}e_2 + \frac{1}{3}e_3$	-0.30	-0.22	0.17
η_{21}	$\frac{2}{3}e_1 + \frac{1}{3}e_2 + \frac{1}{3}e_3 - \frac{1}{3}e_3 - \frac{1}{3}e_1 + \frac{1}{3}e_2 + \frac{1}{3}e_3$	0.36	0.44	0.83
η_{22}	$\frac{2}{3}e_1 + \frac{1}{3}e_2 + \frac{1}{3}e_3 - \frac{1}{3}e_1 + \frac{1}{3}e_3 - \frac{1}{3}e_1 - \frac{1}{3}e_2 + \frac{1}{3}e_3$	0.10	0.33	0.8
η_{23}	$\frac{2}{3}e_1 + \frac{1}{3}e_2 + \frac{1}{3}e_3 + \frac{1}{3}e_3 - \frac{1}{3}e_1 - \frac{1}{3}e_2 + \frac{1}{3}e_3$	0.36	0.44	0.83
η_{24}	$\frac{2}{3}e_1 + \frac{1}{3}e_2 + \frac{1}{3}e_3 - \frac{1}{3}e_1 + \frac{1}{3}e_3 - \frac{1}{3}e_1 - \frac{1}{3}e_2 + \frac{1}{3}e_3$	0.10	0.33	0.8
η_{25}	$\frac{2}{3}e_1 + \frac{1}{3}e_2 + \frac{1}{3}e_3 + -\frac{1}{3} - \frac{1}{3}e_1 - \frac{1}{3}e_2 + \frac{1}{3}e_3$	0	0	0.23
η_{26}	$\frac{2}{3}e_1 + \frac{1}{3}e_2 + \frac{1}{3}e_3 + \frac{1}{3}e_3 - \frac{1}{3}$	0.3	0.22	0.3
η_{27}	$\frac{2}{3}e_1 + \frac{1}{3}e_2 + \frac{1}{3}e_3 - \frac{1}{3}e_1 + \frac{1}{3}e_3 - \frac{1}{3}$	0.03	0.11	0.27
η_{28}	$\frac{2}{3}e_1 + \frac{1}{3}e_2 + \frac{1}{3}e_3 + \frac{1}{3}e_3 - \frac{1}{3}$	0.30	0.22	0.30
η_{29}	$\frac{2}{3}e_1 + \frac{1}{3}e_2 + \frac{1}{3}e_3 - \frac{1}{3}e_1 + \frac{1}{3}e_3 - \frac{1}{3}$	0.03	0.11	0.27
η_{30}	$\frac{2}{3}e_1 + \frac{1}{3}e_2 + \frac{1}{3}e_3 - \frac{1}{3} - \frac{1}{3}$	-0.07	-0.22	-0.3
η_{31}	$-\frac{1}{3} + \frac{1}{3}e_3 - \frac{2}{3}e_1 - \frac{2}{3}e_2 + \frac{1}{3}e_3$	-0.87	-0.55	0.07
η_{32}	$-\frac{1}{3} - \frac{1}{3}e_1 + \frac{1}{3}e_3 - \frac{2}{3}e_1 - \frac{2}{3}e_2 + \frac{1}{3}e_3$	-1.13	-0.67	0.03
η_{33}	$-\frac{1}{3} + \frac{1}{3}e_3 - \frac{2}{3}e_1 - \frac{2}{3}e_2 + \frac{1}{3}e_3$	-0.87	-0.55	0.07
η_{34}	$-\frac{1}{3} - \frac{1}{3}e_1 + \frac{1}{3}e_3 - \frac{2}{3}e_1 - \frac{2}{3}e_2 + \frac{1}{3}e_3$	-1.13	-0.67	0.03
η_{35}	$-\frac{1}{3} - \frac{1}{3} - \frac{2}{3}e_1 - \frac{2}{3}e_2 + \frac{1}{3}e_3$	-1.23	-1	-0.53
η_{36}	$-\frac{1}{3} + \frac{1}{3}e_3 - \frac{1}{3}e_1 - \frac{1}{3}e_2 + \frac{1}{3}e_3$	-0.56	-0.33	0.13
η_{37}	$-\frac{1}{3} - \frac{1}{3}e_1 + \frac{1}{3}e_3 - \frac{1}{3}e_1 - \frac{1}{3}e_2 + \frac{1}{3}e_3$	-0.83	-0.44	0.10
η_{38}	$-\frac{1}{3} + \frac{1}{3}e_3 - \frac{1}{3}e_1 - \frac{1}{3}e_2 + \frac{1}{3}e_3$	-0.56	-0.33	0.13
η_{39}	$-\frac{1}{3} - \frac{1}{3}e_1 + \frac{1}{3}e_3 - \frac{1}{3}e_1 - \frac{1}{3}e_2 + \frac{1}{3}e_3$	-0.83	-0.44	0.10
η_{40}	$-\frac{1}{3} - \frac{1}{3} - \frac{1}{3}e_1 - \frac{1}{3}e_2 + \frac{1}{3}e_3$	-0.93	-0.77	-0.46
η_{41}	$-\frac{1}{3} + \frac{1}{3}e_3 - \frac{1}{3}$	-0.63	-0.55	-0.4
η_{42}	$-\frac{1}{3} - \frac{1}{3}e_1 + \frac{1}{3}e_3 - \frac{1}{3}$	-0.9	-0.67	-0.43
η_{43}	$-\frac{1}{3} + \frac{1}{3}e_3 - \frac{1}{3}$	-0.63	-0.55	-0.4
η_{44}	$-\frac{1}{3} - \frac{1}{3}e_1 + \frac{1}{3}e_3 - \frac{1}{3}$	-0.9	-0.67	-0.43
η_{45}	$-\frac{1}{3} - \frac{1}{3} - \frac{1}{3}$	-1	-1	-1

Table 4.6: Bernoulli principle

Instance	Bernoulli + Maximin	Hodges and Lehmann		
		1	2	3
η_1	$\lambda Bernoulli(\eta_1) + (1-\lambda)(-3)$	-1.32	-1.28	-1.09
η_2	$\lambda Bernoulli(\eta_2) + (1-\lambda)(-2)$	-0.95	-0.84	-0.60
η_3	$\lambda Bernoulli(\eta_3) + (1-\lambda)(-2)$	-0.82	-0.78	-0.59
η_4	$\lambda Bernoulli(\eta_4) + (1-\lambda)(-2)$	-0.95	-0.84	-0.60
η_5	$\lambda Bernoulli(\eta_5) + (1-\lambda)(-2)$	-1	-1	-0.89
η_6	$\lambda Bernoulli(\eta_6) + (1-\lambda)(-3)$	-1.17	-1.17	-1.05
η_7	$\lambda Bernoulli(\eta_7) + (1-\lambda)(-2)$	-0.80	-0.72	-0.57
η_8	$\lambda Bernoulli(\eta_8) + (1-\lambda)(-2)$	-0.67	-0.665	-0.55
η_9	$\lambda Bernoulli(\eta_9) + (1-\lambda)(-1)$	**-0.3**	**-0.22**	**-0.07**
η_{10}	$\lambda Bernoulli(\eta_{10}) + (1-\lambda)(-1)$	-0.35	-0.39	-0.34
η_{11}	$\lambda Bernoulli(\eta_{11}) + (1-\lambda)(-3)$	-1.2	-1.28	-1.34
η_{12}	$\lambda Bernoulli(\eta_{12}) + (1-\lambda)(-2)$	-0.84	-0.84	-0.815
η_{13}	$\lambda Bernoulli(\eta_{13}) + (1-\lambda)(-2)$	-0.7	-0.78	-0.835
η_{14}	$\lambda Bernoulli(\eta_{14}) + (1-\lambda)(-1)$	-0.34	-0.34	-0.34
η_{15}	$\lambda Bernoulli(\eta_{15}) + (1-\lambda)(-1)$	-0.39	-0.50	-0.39
η_{16}	$\lambda Bernoulli(\eta_{16}) + (1-\lambda)(-3)$	-1.47	-1.39	-1.12
η_{17}	$\lambda Bernoulli(\eta_{17}) + (1-\lambda)(-2)$	-1.1	-0.95	-0.64
η_{18}	$\lambda Bernoulli(\eta_{18}) + (1-\lambda)(-2)$	-0.97	-0.89	-0.62
η_{19}	$\lambda Bernoulli(\eta_{19}) + (1-\lambda)(-2)$	-1.1	-0.45	-0.64
η_{20}	$\lambda Bernoulli(\eta_{20}) + (1-\lambda)(-2)$	-1.15	-1.11	-0.92
η_{21}	$\lambda Bernoulli(\eta_{21}) + (1-\lambda)(-3)$	-1.32	-1.28	-1.09
η_{22}	$\lambda Bernoulli(\eta_{22}) + (1-\lambda)(-2)$	-0.95	-0.84	-0.60
η_{23}	$\lambda Bernoulli(\eta_{23}) + (1-\lambda)(-2)$	-0.84	-0.78	-0.59
η_{24}	$\lambda Bernoulli(\eta_{24}) + (1-\lambda)(-1)$	-0.45	-0.34	-0.10
η_{25}	$\lambda Bernoulli(\eta_{25}) + (1-\lambda)(-1)$	-0.50	-0.50	-0.39
η_{26}	$\lambda Bernoulli(\eta_{26}) + (1-\lambda)(-3)$	-1.35	-1.39	-1.35
η_{27}	$\lambda Bernoulli(\eta_{27}) + (1-\lambda)(-2)$	-0.99	-0.95	-0.82
η_{28}	$\lambda Bernoulli(\eta_{28}) + (1-\lambda)(-2)$	-0.85	-0.89	-0.85
η_{29}	$\lambda Bernoulli(\eta_{29}) + (1-\lambda)(-1)$	-0.97	-0.45	-0.32
η_{30}	$\lambda Bernoulli(\eta_{30}) + (1-\lambda)(-1)$	-0.49	-0.61	-0.65
η_{31}	$\lambda Bernoulli(\eta_{31}) + (1-\lambda)(-3)$	-1.94	-1.775	-1.47
η_{32}	$\lambda Bernoulli(\eta_{32}) + (1-\lambda)(-2)$	-1.57	-1.34	-0.985
η_{33}	$\lambda Bernoulli(\eta_{33}) + (1-\lambda)(-2)$	-1.435	-1.28	-0.97
η_{34}	$\lambda Bernoulli(\eta_{34}) + (1-\lambda)(-2)$	-1.57	-1.34	-0.985
η_{35}	$\lambda Bernoulli(\eta_{35}) + (1-\lambda)(-2)$	-1.62	-1.5	-1.27
η_{36}	$\lambda Bernoulli(\eta_{36}) + (1-\lambda)(-3)$	-1.78	-1.67	-1.44
η_{37}	$\lambda Bernoulli(\eta_{37}) + (1-\lambda)(-2)$	-1.415	-1.22	-0.95
η_{38}	$\lambda Bernoulli(\eta_{38}) + (1-\lambda)(-2)$	-1.28	-1.17	-0.94
η_{39}	$\lambda Bernoulli(\eta_{39}) + (1-\lambda)(-1)$	-0.92	-0.72	-0.45
η_{40}	$\lambda Bernoulli(\eta_{40}) + (1-\lambda)(-1)$	-0.97	-0.89	-0.73
η_{41}	$\lambda Bernoulli(\eta_{41}) + (1-\lambda)(-3)$	-1.82	-1.78	-1.7
η_{42}	$\lambda Bernoulli(\eta_{42}) + (1-\lambda)(-2)$	-1.45	-1.34	-1.22
η_{43}	$\lambda Bernoulli(\eta_{43}) + (1-\lambda)(-2)$	- 1.32	-1.28	-1.2
η_{44}	$\lambda Bernoulli(\eta_{44}) + (1-\lambda)(-1)$	-0.95	-0.84	-0.715
η_{45}	$\lambda Bernoulli(\eta_{45}) + (1-\lambda)(-1)$	-1	-1	-1

Table 4.7: Hodges and Lehmann criterion($\lambda = 0.5$)

4.3 Solving Kuhn poker with decision tree extensive form

As the extensive form solution method has already been described in section 3.3, this short section will describe some specific details of this section. It was already explained that the subtrees will be solved as single normal form problems and after the last subtree is solved it will go a step backward to solve the penultimate subtree and the process will be repeated until the entire tree is solved. It is important to add that this method together with different optimality criteria, such as the Maximax, the Maximin, the Laplace and the Hurwicz criteria, can quite easily be seen and solved directly in the decision tree, as these criteria do not need to calculate regret or use some sort of prior information. Therefore, they will be done so directly. For other criteria such as the Minimax Regret, the Bernoulli and the Hodges and Lehmann will have their subtrees solved with the normal form single decision tables and then the corresponding solution branches will be seen in their corresponding tree.

Maximax

In figure 4.2 it is possible to see the extensive form solution with Maximax for the modified Kuhn poker problem. If the extensive form solution were divided into two different normal forms solutions(mainly because of the strategy with the jack), then it would correspond to the normal solutions η_1 and η_6, which are also normal form optimal with the Maximax criterion. It is quite notable how there are a significant lower number of extensive form optimal normal form solutions in comparison to the normal form optimal method(21 normal form optimal solutions). The main reason for this difference between methods is that the extensive form is able to not neglect parts of the tree and it will define different strategies for the different cards that are possible. Another important aspect is that even though the criterion is still quite optimistic, each subtree will be dependent of a reward leaf that actually happens in that subtree. This provides the decision maker to always have an Maximax optimal throught the tree and not just in a specific part of it. Comparing both normal form and extensive form optimal with Maximax criterion, it is difficult to see the big advantages to the normal form method. This is the case mainly because while both of them are optimistic, the normal form will always be dependant of only one reward leaf, independent of how big the decision tree is, while the extensive form is able to take more reward leaves into account and have a strategy that doesn't "forget" any parts of the decision tree.

Maximin criterion

The optimal extensive form solutions with Maximin seen in figure 4.3, if divided into normal form solutions, represent 8 possible normal form solutions, which correspond to: η_9, η_{10}, η_{14}, η_{15}, η_{24}, η_{25}, η_{29} and η_{30}. These are also optimal normal form solutions with Maximin and the main difference between the extensive form and the normal form optimal solutions is that in the scenario, where the king is chosen, the extensive form chooses only b_{raise} and b_{check}, mainly because the minimum of b_{raise} and b_{check} are clearly higher than b_{fold}. This difference occurs because when solving

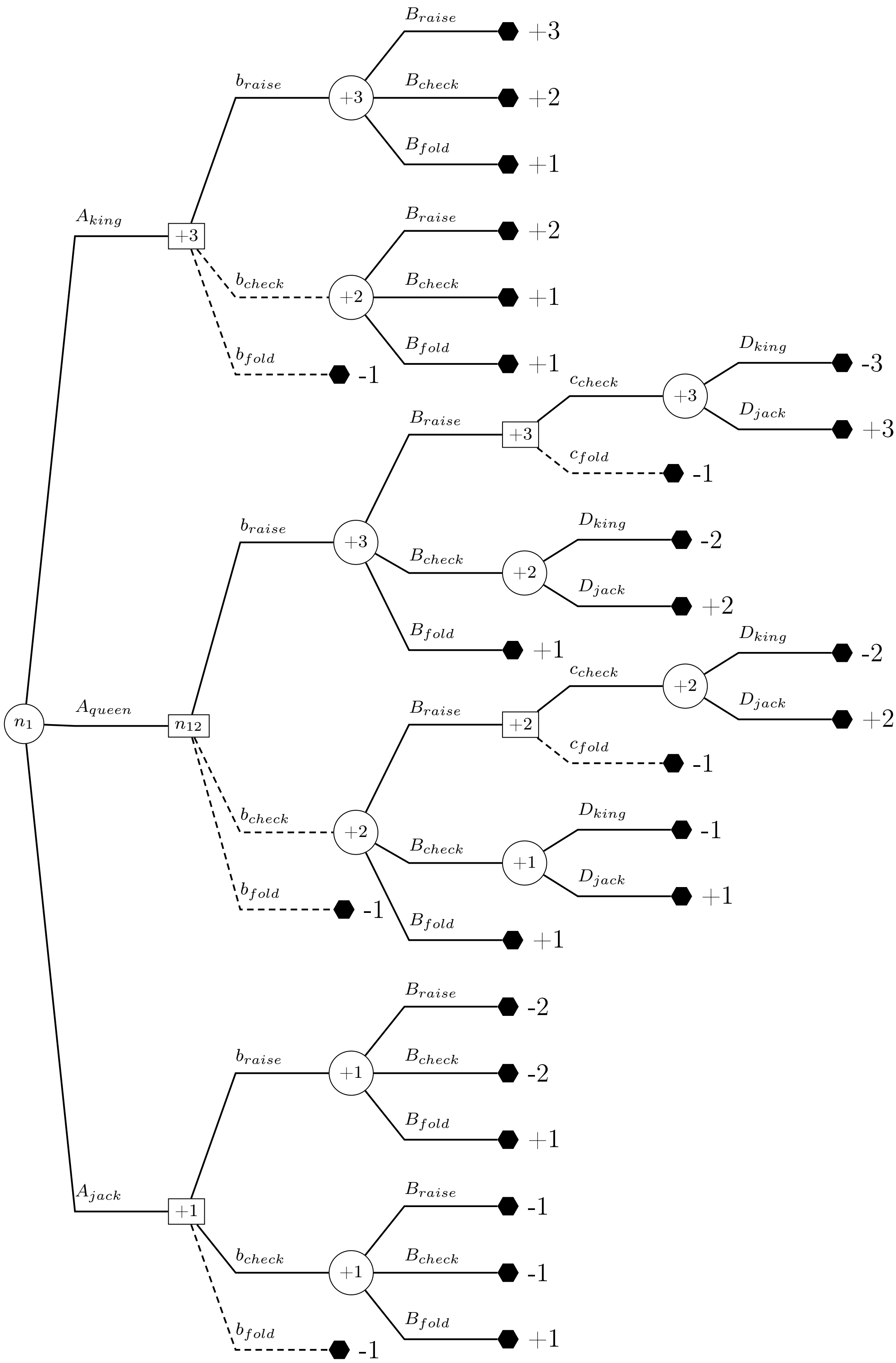

Figure 4.2: Maximax extensive form solution

the tree separately, there are parts that might exclude possible Maximin rewards leaves from the entire decision tree, because in that subtree the Maximin is actually higher than the general Maximin from the tree. This is clearly seen in the A_{king} part of the tree, as the Maximin of this subtree is one and it is higher than from the general tree, which is -1 according to the normal form optimal solutions. The extensive form solution with Maximin is quite interesting in this sense, mainly because it can still be pessimistic, but as the different subtrees are being solved, there might be better Maximins and in turn the strategies might not be as pessimistic as the optimal normal form solution with Maximin.

Laplace criterion

The Laplace extensive form solution seen in figure 4.4, if divided in to normal solutions, would have the same two normal form solutions as the normal form optimal solutions, which are η_6 and η_8. This is not so surprising, as the normal form solution with Laplace criterion tends to be a balanced method as it takes into account all the possible rewards leaves to decide on the normal form optimals. Although this is specially the case in the optimal normal form for our modified Kuhn Poker example, if there were outlier values in some reward leaves, this could cause entire strategies for other parts of the tree to be overlooked, while with the extensive form the solution to other subtrees might not be impacted at all if the subtree with outlier rewards are not directly connected to the other subtrees.

Hurwicz criterion

The extensive form solution with Hurwicz criterion can be seen in figure 4.5. As it was also done for the other extensive form solutions, this result will be transformed into it's corresponding normal form solutions for a better comparison to the normal form optimal solutions to be possible. In this case, the extensive form solution will correspond to the normal form solutions η_6 and η_8. This is the first criterion ,until now, where the corresponding normal form solutions of the extensive form are not normal form optimums for that criterion as well. This might happen because getting the Maximax and Maximin of all the possible rewards together might develop strategies that are based on the different reward leaves of a king and jack(two subtrees that might not relate in a single round as much to one another). While using the extensive form it is possible to individually assess each subtree and their specific maximum and minimun rewards will be separately compared and combined. As these maximums and minimums from the specific subtree often differ to the maximum and minimum reward leaves of the entire tree, it is possible to have quite different strategies and optimal solutions.

It is also interesting that the Hurwicz extensive form solutions and the Laplace extensive form solution are strategically equivalent. There might be different reasons why this is actually the case. The first is that as each chance node in this solution has a maximux of three branches, then getting two of them to represent the outcome is already more than half of the possible paths that can happen if not the entirety of that chance node. The second contributing factor is that the "pessimism" parameter which is being used has also the same consequence as averaging the nodes whenever there are only two chance nodes(multiplying by $\lambda = 0.5$ is the same as dividing by

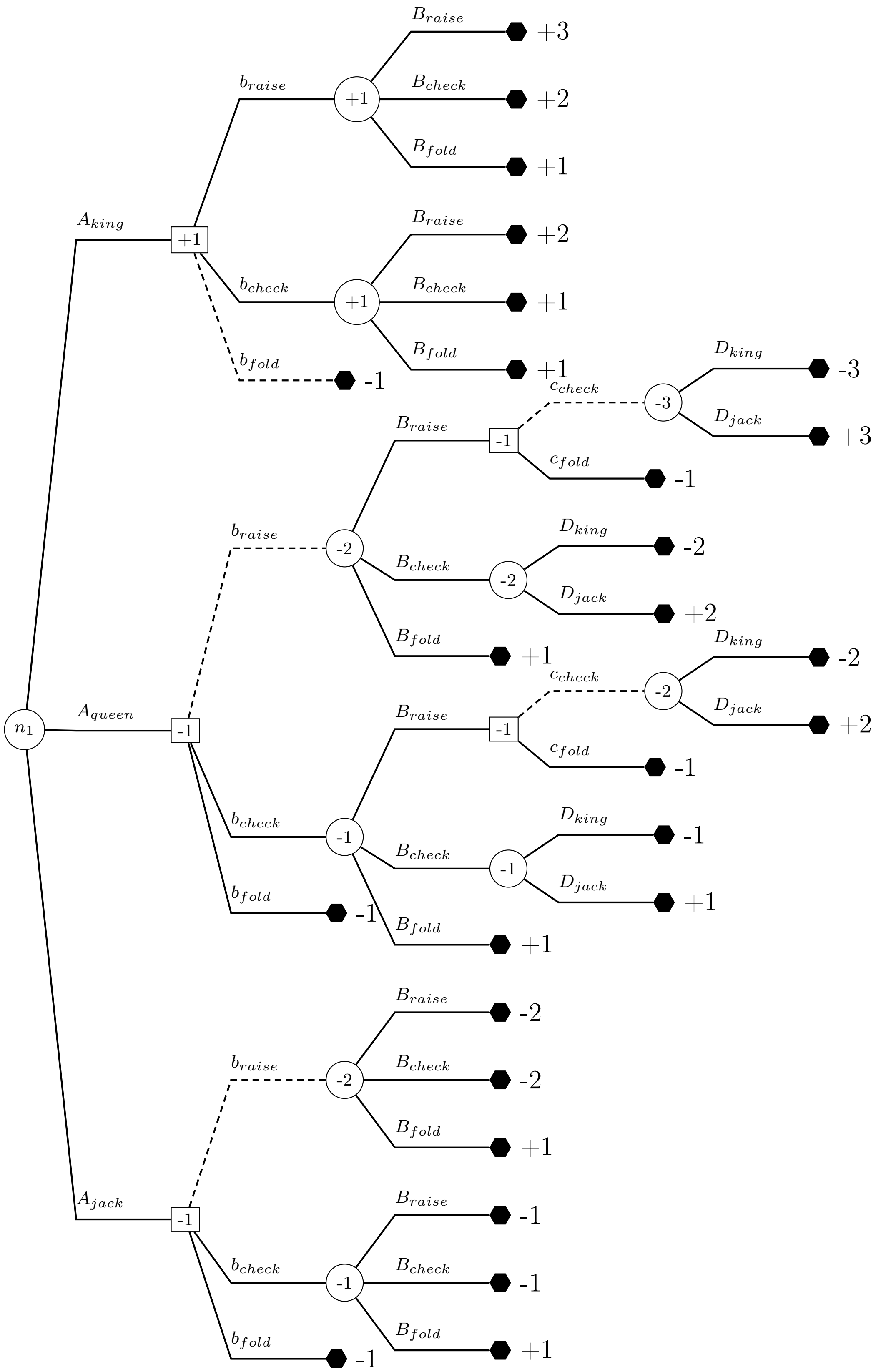

Figure 4.3: Maximin extensive form solution

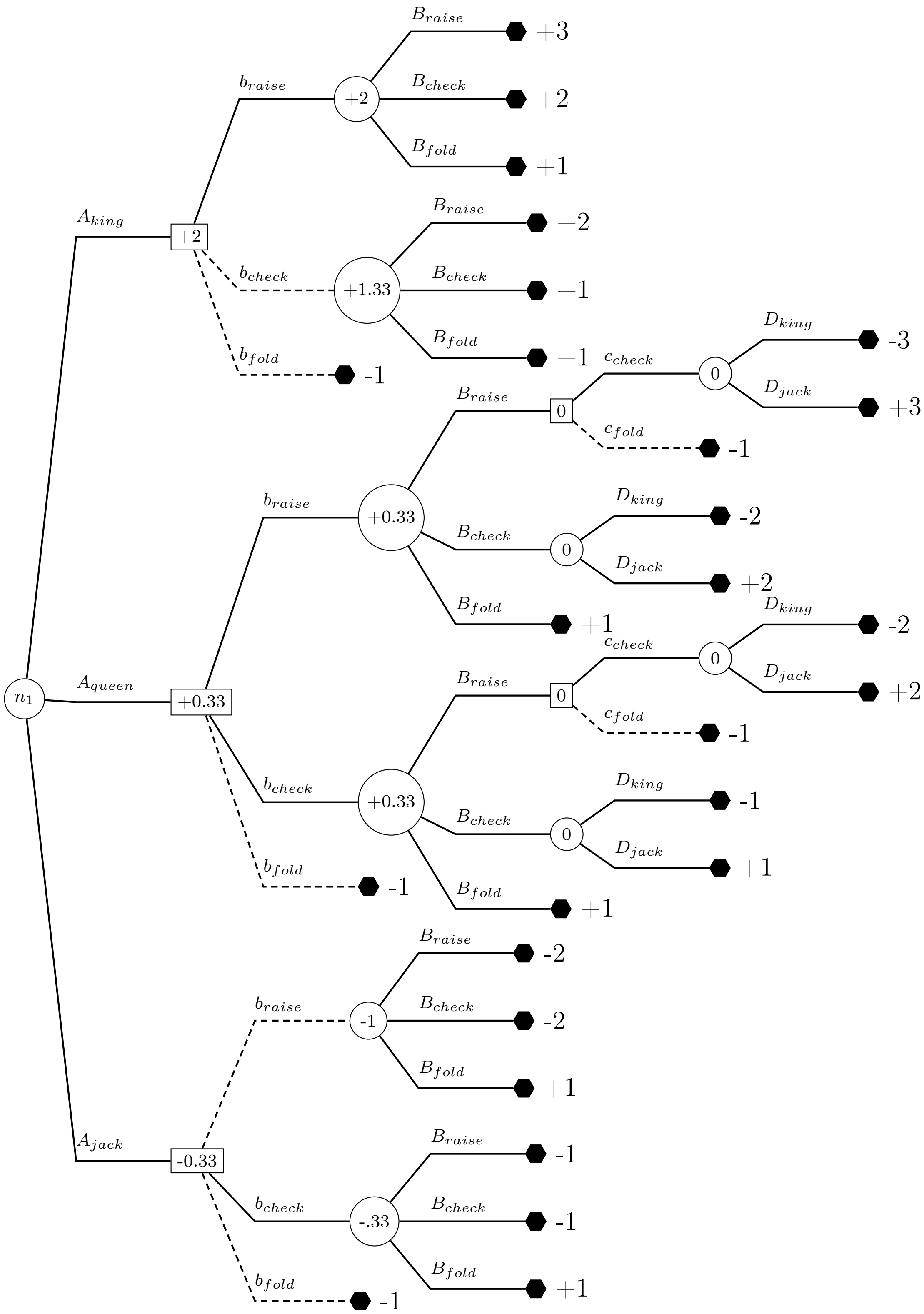

Figure 4.4: Laplace extensive form solution

2). Although the same extensive form solution for both these criteria can be seen in this example, if it is the case that each chance node has a larger number of branches, then the solutions might be completely different, even though to some extent both are balanced criteria for utilities that are not drastically disparate.

Another aspect that has to be discussed are outlier reward leaves, which have a significant bigger or smaller value than all other rewards leaves. It is important to denote that these outliers would for the Hurwicz criterion definitely be more influential to the normal form method, mainly because the entire strategy is based on two reward leaves, while in the extensive form it might only influence parts of the tree and not the entirety of the strategies, as these are defined with a certain degree of independence from one another.

Minimax Regret criterion

To be able to calculate the Minimax regret, first the regret has to be separately calculated, which is represented by the right part of the solution on tables 4.8-4.12. The combination of solutions seen in the tables before mentioned has been concatenated into the decision tree of figure 4.6. This extensive form solution is also a normal form solution and it represents the strategy η_8, which is also a normal form optimum with the Minimax regret. To be able to solve the queen scenario a small possibility space had to be drawn to be able to solve this small normal form problem. In this possibility space $\Omega = \{\omega_1, \omega_2, \omega_3, \omega_4, \omega_5\}$, where $B_{raise} = \{\omega_1, \omega_2\}$, $B_{check} = \{\omega_3, \omega_4\}$, $B_{fold} = \{\omega_5\}$, $c_{check} = \{\omega_1, \omega_3\}$ and $c_{fold} = \{\omega_2, \omega_4\}$.

The problematic with the Minimax regret is similar to Maximax, Maximin and Hurwicz, in the sense that when solving with the extensive form methods the subtrees might have different regrets values than all of the reward leaves being compared together. This explains why there is one normal form solutions to the extensive form optimum and 4 with the normal form method. Still, the normal form optimals are comparing mainly the same nodes, which results in solutions that are able to treat reward leaves individually. On the other hand, outlier rewards leaves would impact the normal form optimal solution more than the extensive, mainly because the entire strategy would be influenced while with the extensive form just parts of the tree that has the outlier reward value.

Acts	States of nature		Regret		Minimax
	D_{king}	D_{jack}	D_{king}	D_{jack}	
c_{check}	-3	3	2	0	**2**
c_{fold}	-1	-1	0	4	4

Table 4.8: Minimax Regret n_{1211}

Acts	States of nature		Regret		Minimax
	D_{king}	D_{jack}	D_{king}	D_{jack}	
c_{check}	-2	2	1	0	**1**
c_{fold}	-1	-1	0	3	3

Table 4.9: Minimax Regret n_{1221}

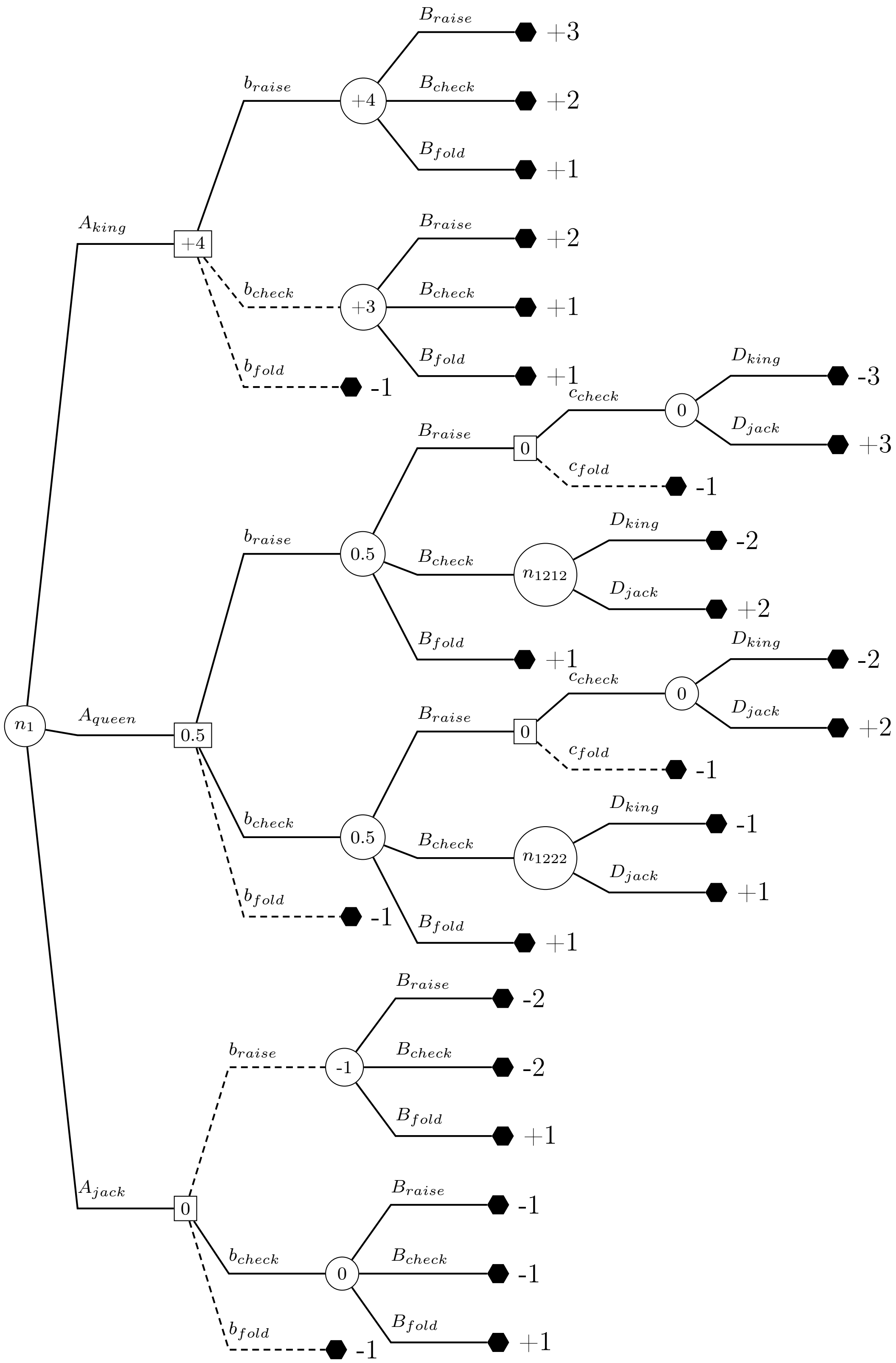

Figure 4.5: Hurwicz extensive form solution

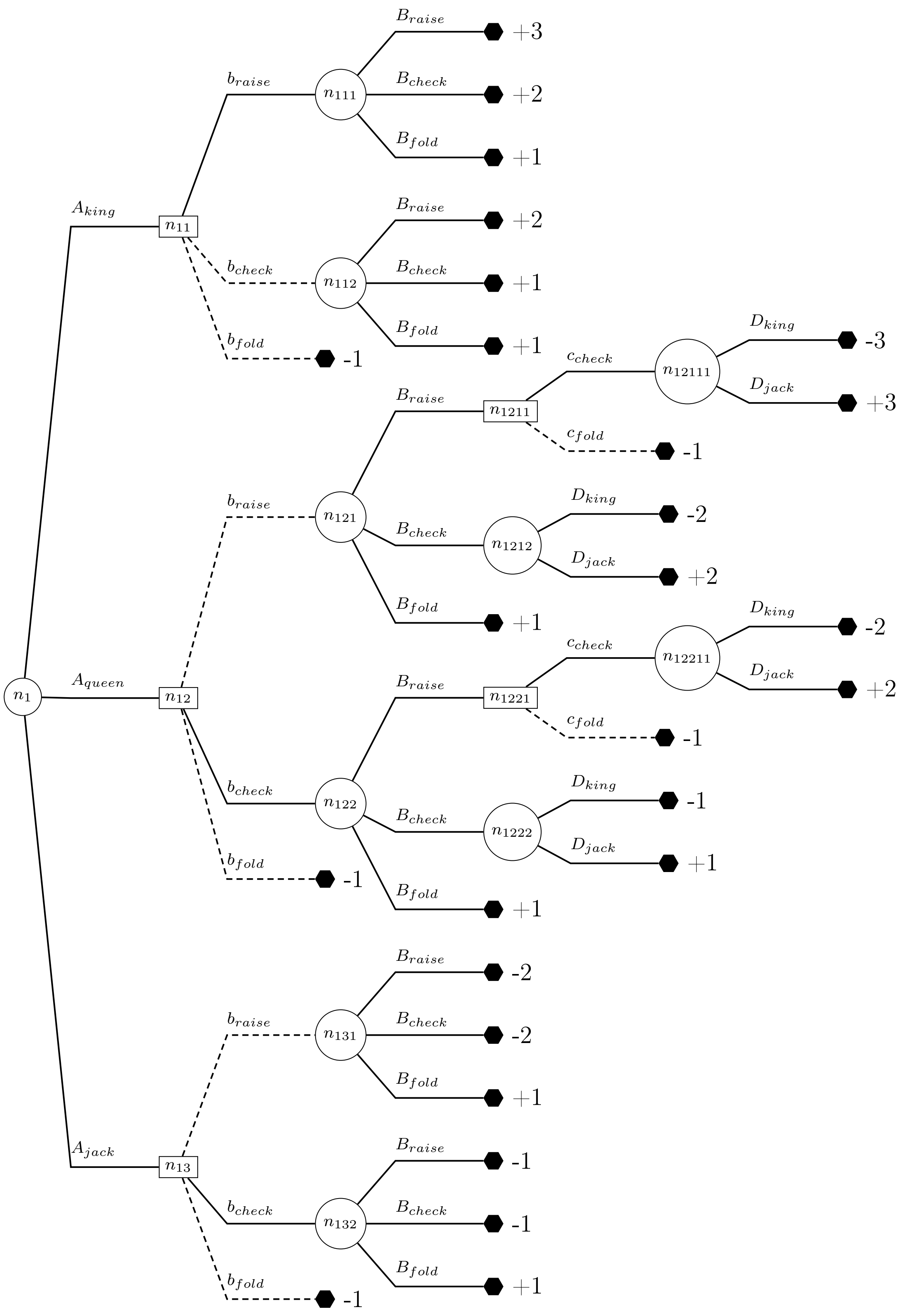

Figure 4.6: Minimax Regret extensive form solution

Acts	States of nature			Regret			Minimax
	B_{raise}	B_{check}	B_{fold}	B_{raise}	B_{check}	$D_{B_{fold}}$	
b_{raise}	3	2	1	0	0	0	**0**
b_{check}	2	1	1	1	1	0	1
b_{fold}	-1	-1	-1	4	3	2	4

Table 4.10: Minimax Regret n_{11}

Acts	States of nature					Regret					Minimax
	ω_1	ω_2	ω_3	ω_4	ω_5	ω_1	ω_2	ω_3	ω_4	ω_5	
b_{raise}	-3	3	-2	2	2	2	0	1	0	0	2
b_{check}	-2	2	-1	1	1	1	1	0	1	1	**1**
b_{fold}	-1	-1	-1	-1	-1	0	4	0	3	3	4

Table 4.11: Minimax Regret n_{12}

Bernoulli principle

Through figure 4.7 it is possible to see the different extensive form solutions with three different combinations of prior probabilities concatenated into the same extensive form solution, while tables 4.13, 4.14 and 4.15 show how the nodes were more specifically solved. It is important to highlight that the combinations of prior probabilities used for the extensive form are the same as the ones used with the normal form method and the notation with c_1, c_2 and c_3 will be continued. The main particularity of this solution is that, even though there are three different prior probabilities that would define drastically different forms of how the opponent plays, in all of them the decisions taken are always the same. This might be the case because the rewards of set actions are better, even though the probabilities could have had an influence on that.

It is also quite notable that the extensive form solution represented as normal form solutions, would correspond exactly to the normal form optimal solution with Bernoulli principle. These solutions would be η_6 and η_8. As it was already described, this strategy makes sense for different reasons on how each part of the strategy acts accordingly to certain aspect of drawing a specific card. Considering the Bernoulli principle is a somewhat balanced criterion and takes into account all the rewards leaves in both normal and extensive methods, it is not that surprising that both methods arrive at the same optimal solution.

Acts	States of nature			Regret			Minimax
	D_{king}	D_{queen}	D_{queen}	D_{king}	D_{king}	D_{queen}	
b_{raise}	-2	-2	1	1	1	0	1
b_{check}	-1	-1	1	0	0	0	**0**
b_{fold}	-1	-1	-1	0	0	2	2

Table 4.12: Minimax Regret n_{13}

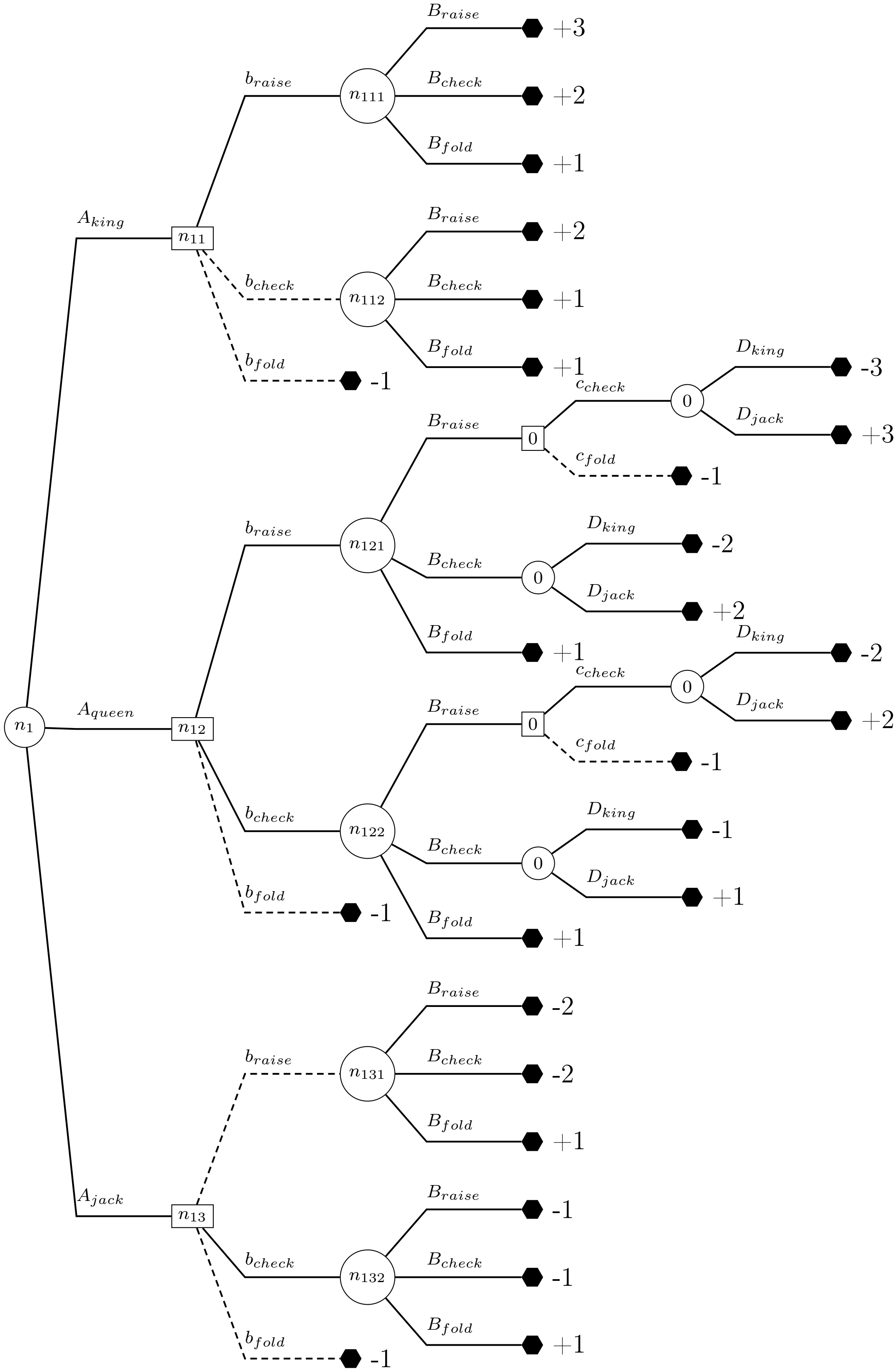

Figure 4.7: Bernoulli principle extensive form solution

Acts	Calculating Bernoulli principle	Bernoulli principle		
		1	2	3
b_{raise}	$3e_1 + 2e_2 + 1e_3$	**3,7**	**2**	**1.3**
b_{check}	$2e_1 + 1e_2 + 1e_3$	1.8	1.33	1.1
b_{fold}	-1	-1	-1	-1

Table 4.13: Bernoulli principle solution for node n_11

Acts	Calculating Bernoulli principle	Bernoulli principle		
		a	b	c
b_{raise}	$0e_1 + 0e_2 + 1e_3$	**0.1**	**0.33**	**0.8**
b_{check}	$0e_1 + 0e_2 + 1e_3$	**0.1**	**0.33**	**0.8**
b_{fold}	-1	-1	-1	-1

Table 4.14: Bernoulli principle solution for node n_12

Hodges and Lehmann criterion

Through figure 4.8 and table 4.16-4.21 it is possible to see the extensive form solution with Hodges and Lehmann criterion. As it was also done with the Bernoulli principle, the Hodges and Lehmann will also use the same combination of probabilities from the normal form with Bernoulli principle and normal form with Hodges and Lehmann criterion. It is interesting how independent of the different combination of prior probabilities of the opponent, they will still induce for all of the combinations the same normal form optimal solutions for the different nodes of the decision tree. One must also denote that the extensive form solution can be divided in to two normal form solutions, which are and η_8 and η_9. η_9 is also the normal form optimal for the normal form with Hodges and Lehmann criterion. The biggest difference with the Bernoulli method is that adding the Maximin makes the criteria somewhat more pessimistic, which is seen by the decision to fold with queen in case the player raises. It is important to denote that in a tree with not so many branches in each chance node adding another Maximin could make it overestimate the lowest bound nodes, as the Bernoulli principle already takes into account the the Maximin utility value in a general form. Still, this higher amount of pessimism could be compensated by the parameter and in situations where the player is overly afraid of the worst possible utility reward it might appear to be quite meaningful.

4.4 Expanding Kuhn Poker

An interesting thought experiment would be to expand the modified Kuhn poker game and talk about how the solutions might differ from the three card version, how

Acts	Calculating Bernoulli principle	Bernoulli principle		
		1	2	3
b_{raise}	$-2e_1 - 2e_2 + 1e_3$	-1.7	-1	0.4
b_{check}	$-1e_1 - 1e_2 + 1e_3$	**-0.8**	**-0.33**	**0.6**
b_{fold}	$-1 - 1 - 1$	-1	-1	-1

Table 4.15: Bernoulli principle solution for node n_13 with a

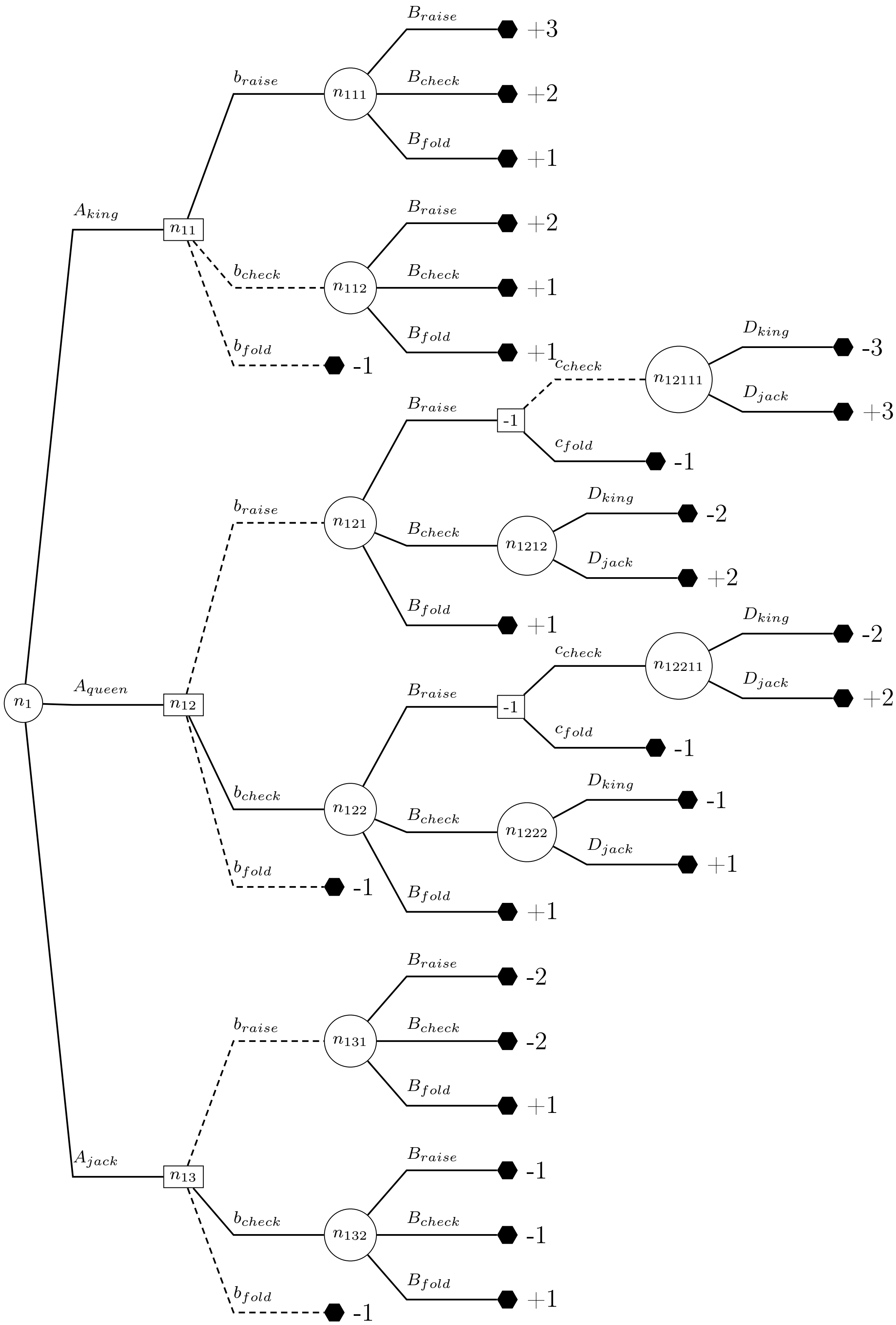

Figure 4.8: Hodges and Lehmann extensive form solution

Acts	Bernoulli + Minimax	Hodges and Lehmann
c_{check}	$\lambda(-3*0.5+3*0.5)+\lambda(-3)$	-1.5
c_{fold}	$\lambda(-1)+\lambda(-1)$	**-1**

Table 4.16: Hodges and Lehmann for node n_{1211}

Acts	Bernoulli + Minimax	Hodges and Lehmann
c_{check}	$\lambda(-2*0.5+2*0.5)+\lambda(-2)$	**-1**
c_{fold}	$\lambda(-1)+\lambda(-1)$	**-1**

Table 4.17: Hodges and Lehman solution for node n_{1221}

Acts	Bernoulli + Minimax	Hodges and Lehmann		
		1	2	3
b_{raise}	$\lambda(3e_1+2e_2+1e_3)+\lambda(1)$	**1.85**	**1.5**	**1.15**
b_{check}	$(2e_1+1e_2+1e_3)+1$	1.4	1.17	1.05
b_{fold}	$\lambda(-1)+\lambda(-1)$	-1	-1	-1

Table 4.18: Hodges and Lehman solution for node n_{11}

Acts	Bernoulli + Minimax	Hodges and Lehmann		
		1	2	3
b_{raise}	$\lambda(-1e_1+(-2*.5+2*0.5)e_2+1e_3)+\lambda(-2)$	-1.35	-1	-0.65
b_{check}	$((-2*0.5+2*0.5)e_1+(-1*0.5+1*0.5)e_2+1e_3)+(-2)$	**-0.95**	**-0.83**	**-0.6**
b_{fold}	$\lambda(-1)+\lambda(-1)$	-1	-1	-1

Table 4.19: Hodges and Lehman criterion solution for node n_{12}

Acts	Bernoulli + Minimax	Hodges and Lehmann		
		1	2	3
b_{raise}	$\lambda(-1e_1+(-2*.5+2*0.5)e_2+1e_3)+\lambda(-2)$	-1.35	-1	-.65
b_{check}	$\lambda((-1)e_1+(-1*0.5+1*0.5)e_2+1e_3)+\lambda(-1)$	**-0.85**	**-0.5**	**-0.15**
b_{fold}	$\lambda(-1)+\lambda(-1)$	-1	-1	-1

Table 4.20: Hodges and Lehman criterion solution for node n_{12}

Acts	Bernoulli + Minimax	Hodges and Lehmann		
		1	2	3
b_{raise}	$\lambda(-2e_1-2e_2+1e_3)+\lambda(-2)$	-1.85	-1-5	-0.8
b_{check}	$\lambda(-1e_1-1e_2+1e_3)+\lambda(-1)$	**-0.9**	**-0.67**	**-0.2**
b_{fold}	$\lambda(-1)+\lambda(-1)$	-1	-1	-1

Table 4.21: Hodges and Lehman solution for node n_{13}

would the different criteria equate with big and small modifications. This idea is quite interesting because the game of poker normally has a significant amount of cards more than the simplified poker version that was introduced. Even though having more cards or solving the real game of texas hold'em poker would have been interesting, the complexity wouldn't be as manageable for this bachelor thesis. If one observes that the modified Kuhn Poker has 45 different possible normal form strategies, it is quite easy to understand the problematic. Therefore, this thought experiment might be an intriguing form of at least thinking how the modified Kuhn poker would equate in a scenario which is closer to the real world.

The first experiment that will be thought about would be what would happen if, instead of having three cards, there were 5 possible cards that can be drawn and the reward leaves were maintained the same. In this case these would be: 9, 10, jack, queen and king. The 10, jack and queen would have the same structure of decision strategies possible as the queen in the three card version, while the king would have the same structure as the king and the 9 would correspond to the structure of the lowest card of the three card version which is the jack. It is quite fair to assume that the scenarios where the decision maker draws the best card(king) and the worst card(9) would be the same strategy wise for most of the different criteria. Mainly because these are exactly the same scenarios where the decision maker will definitely win or definitely lose if the round reaches the end. The biggest changes in this case would probably be with the Bernoulli principle and the Hodges and Lehmann criterion, mainly because the prior probability of cards drawn would be modified. Also, the card in the middle(the jack) would also somewhat be equivalent to the queen scenario in the three card version. The most interesting discussion here would be talking about the 10 and the queen, while for the Minimax, Maximin, Hurwicz and Minimax regret criterion these cards will be solved exactly the same as the queen in the three card scenario, with the other criteria it will differ. This is mainly because the number of reward leaves coming out of the leaves and their rewards being repeated in the last chance node wouldn't influence in anything how the strategy is chosen. But for the Laplace, the Bernoulli and the Hodges and Lehmann criterion the number of rewards leaves with repeated values when winning or losing would change the criterion a bit. One can suppose that in the case of the queen that is a bit better than the jack, there would be somewhat of a tendency to raise more, while with the 10 would be to rather check more.

Although having more cards could change the last chance node for the scenario with 10, jack and queen, it does not necessarily has to change the amount of leaves coming out of the last change node. If instead of having chance nodes that represent the cards from the opponent, such as D_{king}, D_{queen} and D_{jack}, the decision tree might just be structured with D_{win} and D_{lose} and the prior probability would correspond to the probability of the decision maker of winning or losing. This could also be a way of using these methods to solve the last bidding of a texas hold'em poker round. After every card is revealed, there would be one more round of gambling were the decision process could be portrayed exactly like our modified three card Kuhn poker. The main difference would be that instead of having individual cards represented, then certain combinations of the cards would equate to a king, a queen or a jack. As most combinations would have a strategy that assembles more the queen scenario then the king or jack scenarios, then the probability of win for the decision maker would be based on the combination of cards the opponent might have that would win against our decision maker.

4.5 Conclusions

Through the fourth chapter it was possible to see some of the main differences each methods has, while also comparing the different criterion. There are some interesting aspects of the solutions that have to be summarized in this short conclusion. Mostly the combination of of normal form method with criteria such as Maximax and Hurwicz induce answers that are often very dependent of single rewards or rewards that not necessarily will impact the entire strategy, mainly neglecting other parts of the tree. The normal form combined with Maximin doesn't necessarily produce strategies, where the player wins a lot of money but rather doesn't allow him to choose the worst possible reward paths. This doesn't necessarily mean that the utilities chosen are actually a good reward from a perspective of earning the most money. The normal form with Minimax regret is quite intriguing because it compares every action with their equivalent but then chooses based on limiting how bad a regret can be. This in itself is not that different from the Maximin and it is also very dependent of just one action with a specific state of nature after the regret has been calculated. The normal form with Laplace, Bernoulli or Hodges and Lehmann criteria try to solve the decision tree taking into account all the reward leaves, which produced a smaller number of normal form optimal solutions but at the same time didn't seem to neglect other parts of the tree.

The extensive form method with the different criteria obviously maintained the major characteristics of each criteria, still it was able to treat the different parts of the trees as individual trees, which would combine to make strategies that don't neglect certain parts of the trees or that are too focused in only one or two rewards leaves. It was also quite interesting how only the Hurwicz and the Hodges and Lehmann criteria had extensive form solutions, which had normal form solution which weren't also normal form optimal solution, while all the other has solutions which were normal form optimal.

One interesting aspect of the research which could still be talked about in the future is how the combination of different criteria could compliment one another throughout more rounds of the game being played. Another important aspect would be how could neural networks, with the objective of playing poker, be developed based on different optimality criterion game and how they would use these different criterion to learn how to play in a manner which maximizes their wins in every different situation, if they would use one criteria more than the other and other different possible usages.

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